A Little Challenge Problem for Real Number Theorem Proving

Rob Arthan

Lemma 1 Ltd.

Based on joint work with Robert M. Solovay and John Harrison

Workshop on Interactive Theorem Proving, Cambridge 24th August 2009

Decidability for theories of real arithmetic

- Truth in the first order language $\mathcal{L}_R$ of $\mathbb{R}$ is decidable (Tarski).
- If $N(x) \Leftrightarrow x \in \mathbb{N}$, then truth in $\mathcal{L}_R + N$ is undecidable since if $Q(x_1, \ldots, x_k)$ is a quantifier-free formula of natural number arithmetic

$$\mathbb{N} \models \exists x_1 \ldots x_n \cdot Q(x_1, \ldots, x_n) \qquad \textit{(i)}$$

iff

$$\mathbb{R} \models \exists x_1 \ldots x_n \cdot N(x_1) \wedge \ldots N(x_n) \wedge Q(x_1, \ldots, x_n) \qquad \textit{(ii)}$$

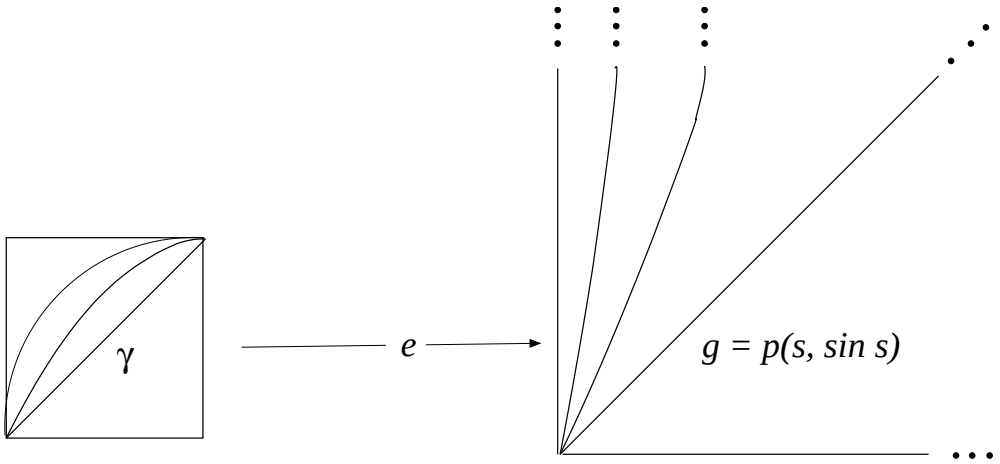
and the truth of *(i)* is undecidable in general (Matiyasevich).

- E.g., truth in $\mathcal{L}_R + \mathsf{sin}$ is undecidable since $N(x) \Leftrightarrow x \in \mathbb{N}$ where

$$N(x) := x \geq 0 \wedge \exists \pi \cdot 0 < \pi < 4 \wedge \mathsf{sin}\ \pi = \mathsf{sin}\ x\pi = 0$$

- What other f make truth in $\mathcal{L}_R + f$ undecidable?

Encoding sin in a bounded concave γ. I

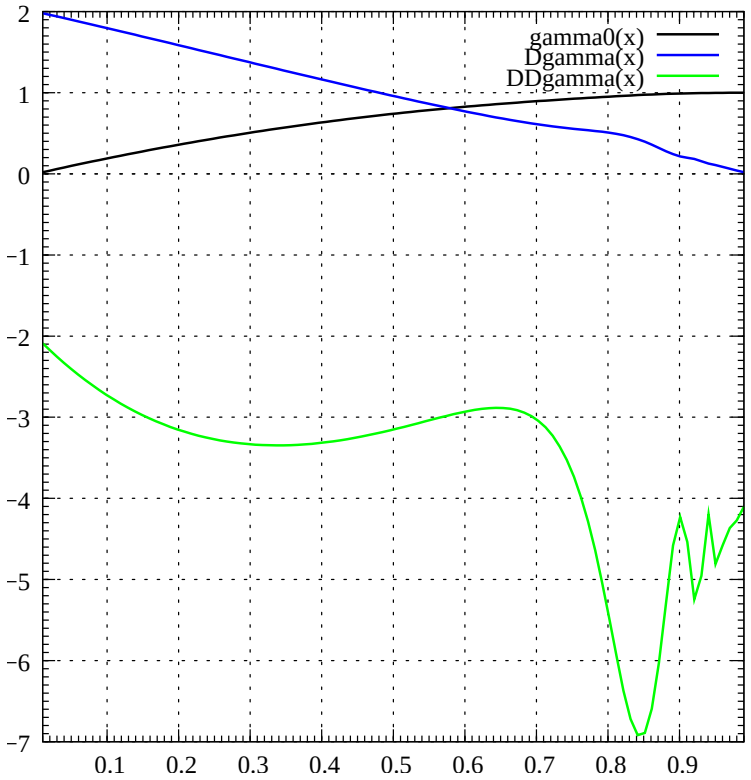


$$
\begin{array}{lrcl}
 & e(x,y) & = & (\frac{x}{1-x}, \frac{y}{1-y}) \\
\text{so} & \gamma(x) & = & q(g(p(x))) \\
\text{where} & p(x) & = & \frac{x}{1-x}, \quad q(s) = \frac{s}{1+s} \\
\text{and} & g(s) & = & Ks + s^2 + \frac{1}{M}\sin s
\end{array}
$$

Question: do any K and M make γ concave?

Encoding sin in a bounded concave γ. II

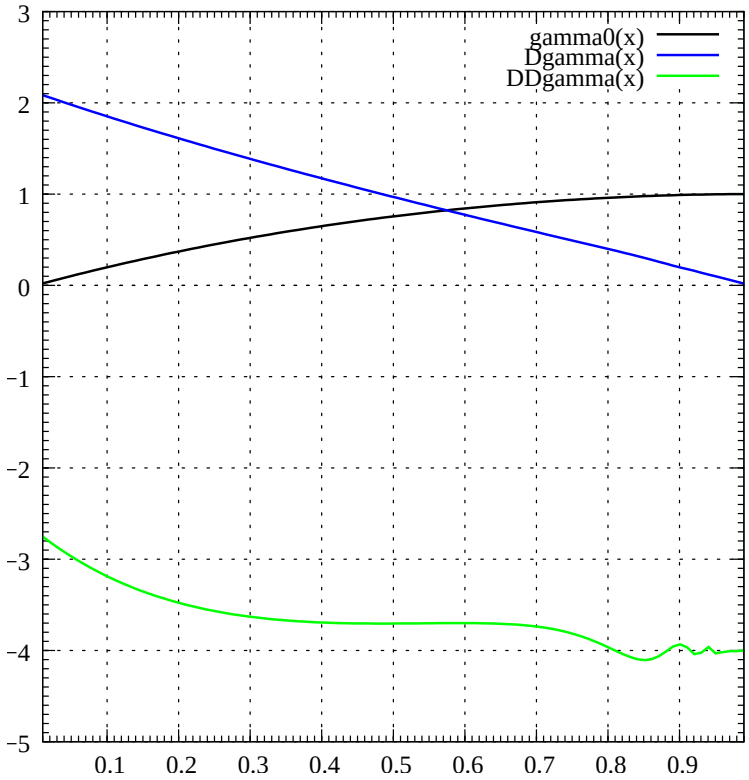
Answer: yes! Here are γ, γ' and γ'' for $K = M = 1$:



Can you **prove** $\gamma'' < 0$?

Encoding sin in a bounded concave γ. III

Answer: ah! well, I can prove it for $K = 2$ and $M \geq 9$:



Challenge: is there any automation to help out there?

The challenge problem as a trigonometric polynomial

- γ is a trigonometric rational function.
- If trigonometric polynomials are more to your taste, put $s = \frac{x}{1+x}$, and then $\gamma''(x)$ has the same sign as $h(s)$ where

$$\begin{aligned} h(s) \quad = \quad & (1 + Ks + s^2 + \tfrac{\sin s}{M})[(1+s)(2 - \tfrac{\sin s}{M}) + 2K + 4s + \tfrac{2\cos s}{M}] \\ & - 2(1+s)(K + 2s + \tfrac{\cos s}{M})^2. \end{aligned}$$

- If you forget the formulas, you can find these slides at

  `http://www.lemma-one.com/papers/papers.html`

Remarks

- The classical method for this kind of problem is to find polynomial bounds, subdividing the interval if necessary.
- Happily, we were free to use more convenient values for K and M.
- Some types of problem should be well suited to automation . . .
- . . . but what are the useful decidable fragments?
- Paulson's MetiTarski system is very promising, but didn't solve the little challenge "out of the box".