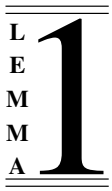

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Theorems on Finiteness (Draft)

Abstract

This paper supplies proofs of some theorems about finite sets. It is part of work in progress on ProofPower-HOL.

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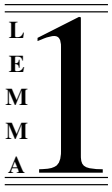

CONTENTS

1 INTRODUCTION

This document contains some proofs which are mainly concerned with finiteness as defined in the theory *fin_set* supplied as part of the ProofPower-HOL system. Some miscellaneous material about certain other concepts defined in the ancestors of *fin_set* is also provided.

The material displays several aspects of the use of ProofPower-HOL., including:

1. Proof of an induction theorem and its use to produce an induction tactic.
2. Proof of a theorem for use as a rewrite rule to "compute" values of a function in certain useful special cases (namely the minimum function *Min* applied to set displays whose elements are numeric constants).

Note that the proofs are typically packaged in the following style:

Example

```
val blah_blah_thm = save_thm("fin_set_induction_thm", (
push_goal([], (* Statement of theorem *));
(* Rest of subgoal package command script goes here *)
pop_thm()
));
```

To experiment with such a proof interactively simply omit the first line. I.e. cut-and-paste the *push_goal* command into the ProofPower-HOL window followed by the subgoal package commands (which are almost invariably just tactic applications of the form $a(\ldots)$) modifying these as you wish. To finish the proof off either cut-and-paste the *pop_thm* line followed by a semi-colon (if you have completed the proof) or abandon the proof attempt with *drop_main_goal*.

2 PREAMBLE

SML

```
open_theory"fin_set";
new_theory"fin_thms";
set_pc"hol";
```

3 THE THEOREMS

3.1 Induction over Finite Sets

SML

```
val fin_set_induction_thm = save_thm("fin_set_induction_thm", (
push_goal([], ⌜
                ∀P•    P {} ∧ (∀a x•a ∈ Finite ∧ P a ∧ ¬x ∈ a ⇒ P({x} ∪ a))
                ⇒      ∀a•a ∈ Finite ⇒ P a
⌝);
a(REPEAT strip_tac);
a(asm_ante_tac⌜a ∈ Finite⌝);
a(rewrite_tac[get_spec⌜Finite⌝]);
a(REPEAT strip_tac);
a(spec_nth_asm_tac 1 ⌜{b | b ∈ Finite ∧ P b}⌝);
(* *** Goal "1" *** *)
a(swap_asm_concl_tac ⌜¬ {} ∈ Finite⌝);
a(rewrite_tac[get_spec⌜Finite⌝]);
a(PC_T "hol1" (REPEAT strip_tac));
(* *** Goal "2" *** *)
a(swap_asm_concl_tac ⌜a' ∈ Finite⌝);
a(asm_ante_tac ⌜¬ {x} ∪ a' ∈ Finite⌝);
a(rewrite_tac[get_spec⌜Finite⌝]);
a(PC_T "hol1" (contr_tac));
a(spec_nth_asm_tac 1 ⌜s⌝);
(* *** Goal "2.1" *** *)
a(list_spec_nth_asm_tac 5 [⌜a''⌝, ⌜x'⌝]);
(* *** Goal "2.2" *** *)
a(list_spec_nth_asm_tac 4 [⌜a'⌝, ⌜x⌝]);
(* *** Goal "3" *** *)
a(swap_asm_concl_tac ⌜¬ P ({x} ∪ a')⌝);
a(cases_tac⌜x ∈ a'⌝);
(* *** Goal "3.1" *** *)
a(LEMMA_T⌜{x} ∪ a' = a'⌝ asm_rewrite_thm_tac);
a(PC_T "hol1" (REPEAT strip_tac));
a(asm_rewrite_tac[]);
(* *** Goal "3.2" *** *)
a(list_spec_nth_asm_tac 6 [⌜a'⌝, ⌜x⌝]);
pop_thm()
));
```

SML
```
val fin_set_induction_tac : TACTIC =
                gen_induction_tac1 fin_set_induction_thm;
```

3.2 Theorems about *Min*

SML
```
val min_thm = save_thm("min_thm", (
push_goal([], ⌜∀m a•m ∈ a ⇒ Min a ∈ a ∧ Min a ≤ m⌝);
a(REPEAT ∀_tac THEN ⇒_tac);
a(strip_asm_tac (rewrite_rule[](∀_elim⌜(λj•j ∈ a)⌝≤_well_order_thm))
        THEN asm_prove_tac[]);
(* *** Goal "1" *** *)
a(strip_asm_tac (list_∀_elim[⌜m'⌝, ⌜a⌝](get_spec⌜Min⌝)));
(* *** Goal "1.1" *** *)
a(asm_prove_tac[]);
(* *** Goal "1.2" *** *)
a(asm_rewrite_tac[] THEN REPEAT strip_tac);
(* *** Goal "2" *** *)
a(strip_asm_tac (list_∀_elim[⌜m'⌝, ⌜a⌝](get_spec⌜Min⌝)));
(* *** Goal "2.1" *** *)
a(asm_prove_tac[]);
(* *** Goal "2.2" *** *)
a(asm_rewrite_tac[] THEN asm_prove_tac[]);
pop_thm()
));
```

3.3 Theorems about *Min*

SML
```
val min_clause1 = (
push_goal([], ⌜∀m•Min {m} = m⌝);
a(REPEAT strip_tac);
a(strip_asm_tac (list_∀_elim[⌜m⌝, ⌜{m}⌝](get_spec⌜Min⌝)));
a(swap_asm_concl_tac⌜¬m ≤ i⌝ THEN asm_rewrite_tac[≤_clauses]);
pop_thm()
);
```

SML

```
val min_clause2 = (
push_goal([], ⌜∀m n•Min {m; n} = if m ≤ n then m else n⌝);
a(REPEAT strip_tac);
a(strip_asm_tac (list_∀_elim[⌜if m ≤ n then m else n⌝, ⌜{m; n}⌝](get_spec⌜Min⌝)));
(* *** Goal "1" *** *)
a(all_asm_ante_tac THEN cases_tac⌜m ≤ n⌝ THEN asm_rewrite_tac[]);
(* *** Goal "2" *** *)
a(swap_asm_concl_tac⌜¬(if m ≤ n then m else n) ≤ i⌝ THEN asm_rewrite_tac[]);
a(cases_tac⌜m ≤ n⌝ THEN asm_rewrite_tac[≤_clauses]);
a(strip_asm_tac(list_∀_elim[⌜m⌝, ⌜n⌝]≤_cases_thm));
(* *** Goal "3" *** *)
a(swap_asm_concl_tac⌜¬(if m ≤ n then m else n) ≤ i⌝ THEN asm_rewrite_tac[]);
a(cases_tac⌜m ≤ n⌝ THEN asm_rewrite_tac[≤_clauses]);
pop_thm()
);
```

SML

```
val min_clause3 = (
push_goal([], ⌜∀m a•Min ({m} ∪ a) = if a = {} then m else Min{m; Min a}⌝);
a(REPEAT strip_tac);
a(cases_tac⌜a = {}⌝ THEN asm_rewrite_tac[]);
(* *** Goal "1" *** *)
a(rewrite_tac[min_clause1]);
(* *** Goal "2" *** *)
a(all_asm_ante_tac THEN (PC_T"sets_ext" strip_tac));
a(all_asm_ante_tac THEN strip_tac);
a(strip_asm_tac (list_∀_elim[⌜x⌝, ⌜a⌝]min_thm));
a(lemma_tac⌜∀ i• i ∈ {m} ∪ a ⇒ Min {m; Min a} ≤ i⌝);
(* *** Goal "2.1" *** *)
a(REPEAT strip_tac THEN rewrite_tac[min_clause2]);
(* *** Goal "2.1.1" *** *)
a(cases_tac⌜m ≤ Min a⌝ THEN asm_rewrite_tac[≤_clauses]);
a(strip_asm_tac (list_∀_elim[⌜m⌝, ⌜Min a⌝]≤_cases_thm));
(* *** Goal "2.1.2" *** *)
a(strip_asm_tac (list_∀_elim[⌜i⌝, ⌜a⌝]min_thm));
a(cases_tac ⌜m ≤ Min a⌝ THEN asm_rewrite_tac[]);
a(fc_tac [≤_trans_thm] THEN asm_fc_tac[]);
(* *** Goal "2.2" *** *)
a(LEMMA_T ⌜Min {m; Min a} ∈ {m} ∪ a⌝ asm_tac);
(* *** Goal "2.2.1" *** *)
a(cases_tac⌜m ≤ Min a⌝ THEN asm_rewrite_tac[min_clause2] THEN REPEAT strip_tac);
(* *** Goal "2.2.2" *** *)
```

```
a(ante_tac (list_∀_elim[⌜Min {m; Min a}⌝, ⌜({m} ∪ a)⌝](get_spec⌜Min⌝))
        THEN asm_rewrite_tac[]);
pop_thm()
);
```

SML

```
val min_clauses = save_thm("min_clauses",
        list_∧_intro[min_clause1, min_clause2, min_clause3]);
```

3.4 Relationship between *Finite* and *Elems*

SML

```
val fin_set_thm1 = save_thm("fin_set_thm1", (
push_goal([], ⌜∀a• a ∈ Finite ⇒ ∃list•a = Elems list ∧ list ∈ Distinct⌝);
a(strip_tac);
a fin_set_induction_tac;
(* *** Goal "1" *** *)
a(∃_tac ⌜[]⌝ THEN rewrite_tac(map get_spec[⌜Elems⌝, ⌜Distinct⌝]));
(* *** Goal "2" *** *)
a(∃_tac ⌜Cons x list⌝ THEN asm_rewrite_tac(map get_spec[⌜Elems⌝, ⌜Distinct⌝]));
a(asm_ante_tac ⌜¬ x ∈ a⌝ THEN asm_rewrite_tac[]);
pop_thm()
));
```

SML

```
val elems_thm1 = save_thm("elems_thm1", (
push_goal([], ⌜∀list• Elems list = {} ⇔ list = []⌝);
a(strip_tac THEN strip_asm_tac(∀_elim⌜list:'a LIST⌝ list_cases_thm));
(* *** Goal "1" *** *)
a(asm_rewrite_tac[get_spec⌜Elems⌝]);
(* *** Goal "2" *** *)
a(asm_rewrite_tac[get_spec⌜Elems⌝]);
a(PC_T "sets_ext" (REPEAT strip_tac));
a(REPEAT strip_tac);
a(∃_tac⌜x⌝ THEN REPEAT strip_tac);
pop_thm()
));
```

SML

```
val elems_thm2 = save_thm("elems_thm2", (
push_goal([], ⌜(∀list• Elems list = {} ⇔ list = [])
        ∧ (∀list• {} = Elems list ⇔ list = [])⌝);
a(rewrite_tac[elems_thm1]);
a(rewrite_tac[conv_rule (ONCE_MAP_C eq_sym_conv) elems_thm1]);
a(REPEAT strip_tac THEN asm_rewrite_tac[]);
pop_thm()
));
```

SML

```
val elems_thm3 = save_thm("elems_thm3", (
push_goal([], ⌜∀list1 list2• Elems(list1 @ list2) = Elems list1 ∪ Elems list2⌝);
a(REPEAT strip_tac);
a(list_induction_tac⌜list1⌝);
(* *** Goal "1" *** *)
a(rewrite_tac(map get_spec[⌜Append⌝, ⌜Elems⌝]));
(* *** Goal "2" *** *)
a(asm_rewrite_tac(map get_spec[⌜Append⌝, ⌜Elems⌝]));
a(PC_T"hol1" (REPEAT strip_tac));
pop_thm()
));
```

SML

```
val length_thm = save_thm("length_thm", (
push_goal([], ⌜∀list1 list2• Length (list1 @ list2) = Length list1 + Length list2⌝);
a(REPEAT strip_tac);
a(list_induction_tac⌜list1⌝ THEN rewrite_tac(map get_spec[⌜Length⌝, ⌜Append⌝]));
a(asm_rewrite_tac[plus_assoc_thm]);
pop_thm()
));
```

SML

```
val ↾_thm1 = save_thm("↾_thm1", (
push_goal([], ⌜∀list a•Elems(list ↾ a) = Elems list ∩ a⌝);
a(REPEAT strip_tac);
a(list_induction_tac⌜list⌝ THEN rewrite_tac(map get_spec[⌜Elems⌝, ⌜$↾⌝]));
a(strip_tac THEN cases_tac ⌜x ∈ a⌝ THEN asm_rewrite_tac[]);
(* *** Goal "1" *** *)
a(asm_rewrite_tac[get_spec⌜Elems⌝]);
a(PC_T "hol1" (REPEAT strip_tac));
a(asm_rewrite_tac[]);
(* *** Goal "2" *** *)
```

```
a(PC_T "hol1" (REPEAT strip_tac));
a(asm_ante_tac⌜x' ∈ a⌝ THEN asm_rewrite_tac[]);
pop_thm()
));
```

SML

```
val ↾_thm2 = save_thm("↾_thm2", (
push_goal([], ⌜∀list; a:'a SET•Length((list ↾ a) @ (list ↾ ∼a)) = Length list⌝);
a(REPEAT strip_tac);
a(list_induction_tac⌜list⌝ THEN rewrite_tac(map get_spec[⌜Length⌝, ⌜$↾⌝, ⌜Append⌝]));
a(strip_tac THEN cases_tac ⌜x ∈ a⌝ THEN asm_rewrite_tac[]);
(* *** Goal "1" *** *)
a(asm_rewrite_tac(map get_spec[⌜Length⌝, ⌜$↾⌝, ⌜Append⌝]));
(* *** Goal "2" *** *)
a(all_asm_ante_tac THEN rewrite_tac[length_thm] THEN REPEAT strip_tac);
a(asm_rewrite_tac[get_spec⌜Length⌝, ∀_elim⌜1⌝plus_order_thm]);
pop_thm()
));
```

SML

```
val ↾_thm3 = save_thm("↾_thm3", (
push_goal([], ⌜∀list a•Elems list ⊆ a ⇒ list ↾ a = list ⌝);
a(strip_tac);
a(list_induction_tac⌜list⌝);
(* *** Goal "1" *** *)
a(rewrite_tac[get_spec⌜$↾⌝]);
(* *** Goal "2" *** *)
a(asm_rewrite_tac(map get_spec[⌜$↾⌝, ⌜Elems⌝]));
a(REPEAT strip_tac);
a(lemma_tac ⌜x ∈ a⌝);
(* *** Goal "2.1" *** *)
a(POP_ASM_T ante_tac);
a(PC_T"hol1"(REPEAT strip_tac));
a(spec_nth_asm_tac 1 ⌜x⌝);
(* *** Goal "2.2" *** *)
a(asm_rewrite_tac[]);
a(lemma_tac ⌜Elems list ⊆ a⌝);
(* *** Goal "2.2.1" *** *)
a(asm_ante_tac⌜{x} ∪ Elems list ⊆ a⌝ THEN PC_T"hol1"(REPEAT strip_tac));
a(spec_nth_asm_tac 2 ⌜x'⌝);
(* *** Goal "2.2.2" *** *)
a(spec_nth_asm_tac 4 ⌜a⌝);
pop_thm()
```

```
));
```

SML

```
val ↾_thm4 = save_thm("↾_thm4", (
push_goal([], ⌜∀list; x:'a•x ∈ Elems list ⇒ Length(list ↾ ∼{x}) < Length list ⌝);
a(strip_tac);
a(list_induction_tac⌜list⌝ THEN asm_rewrite_tac[get_spec⌜Elems⌝]);
(* *** Goal "" *** *)
a(asm_rewrite_tac(map get_spec[⌜$↾⌝, ⌜Length⌝]) THEN REPEAT strip_tac);
(* *** Goal "1" *** *)
a(asm_rewrite_tac[get_spec⌜Length⌝] THEN cases_tac⌜x ∈ Elems list⌝);
(* *** Goal "1.1" *** *)
a(spec_nth_asm_tac 3 ⌜x⌝);
a(strip_asm_tac(list_∀_elim
        [⌜Length (list ↾ ∼ {x})⌝, ⌜Length list⌝, ⌜Length list + 1⌝]
        less_trans_thm));
a(lemma_tac⌜Elems list ⊆ ∼{x}⌝);
(* *** Goal "1.2.1" *** *)
a(PC_T"hol1"(REPEAT strip_tac THEN rewrite_tac[]));
a(swap_asm_concl_tac ⌜x'' ∈ Elems list⌝ THEN asm_rewrite_tac[]);
(* *** Goal "1.2.2" *** *)
a(strip_asm_tac(list_∀_elim[⌜list⌝, ⌜∼{x}⌝]↾_thm3));
a(asm_rewrite_tac[]);
(* *** Goal "2" *** *)
a(DROP_NTH_ASM_T 2 (strip_asm_tac o ∀_elim ⌜x'⌝));
a(cases_tac⌜x = x'⌝ THEN asm_rewrite_tac[]);
(* *** Goal "2.1" *** *)
a(strip_asm_tac(list_∀_elim
        [⌜Length (list ↾ ∼ {x'})⌝, ⌜Length list⌝, ⌜Length list + 1⌝]
        less_trans_thm));
(* *** Goal "2.2" *** *)
a(rewrite_tac[get_spec⌜Length⌝] THEN REPEAT strip_tac);
pop_thm()
));
```

SML

```
val distinct_thm1 = save_thm("distinct_thm1", (
push_goal([], ⌜∀list1 list2•
                list1 ∈ Distinct ∧ Elems list1 = Elems list2
        ⇒       Length list1 ≤ Length list2⌝);
a(strip_tac);
a(list_induction_tac⌜list1⌝);
(* *** Goal "1" *** *)
```

```
a(rewrite_tac(elems_thm2 :: map get_spec[⌜Elems⌝, ⌜Distinct⌝, ⌜Length⌝]));
(* *** Goal "2" *** *)
a(REPEAT strip_tac);
a(lemma_tac ⌜Length ([x] @ (list2 ↑ ∼{x})) ≤ Length list2⌝);
(* *** Goal "2.1" *** *)
a(rewrite_tac[get_spec⌜Length⌝, length_thm]);
a(cases_tac⌜x ∈ Elems list2⌝);
(* *** Goal "2.1.1" *** *)
a(strip_asm_tac(list_∀_elim[⌜list2⌝, ⌜x⌝]↑_thm4));
a(asm_ante_tac⌜Length (list2 ↑ ∼ {x}) < Length list2⌝ THEN
        rewrite_tac[∀_elim⌜1⌝plus_order_thm, less_def]);
(* *** Goal "2.1.2" *** *)
a(lemma_tac⌜Elems list2 ⊆ ∼{x}⌝);
(* *** Goal "2.1.2.1" *** *)
a(PC_T"hol1"(REPEAT strip_tac THEN rewrite_tac[]));
a(swap_asm_concl_tac⌜x′ ∈ Elems list2⌝ THEN asm_rewrite_tac[]);
(* *** Goal "2.1.2.2" *** *)
a(lemma_tac⌜x ∈ Elems list2⌝);
a(asm_ante_tac ⌜Elems (Cons x list1) = Elems list2⌝);
a(rewrite_tac[get_spec⌜Elems⌝]);
a(PC_T"hol1"(REPEAT strip_tac));
a(spec_nth_asm_tac 1⌜x⌝);
(* *** Goal "2.2" *** *)
a(DROP_NTH_ASM_T 3 (strip_asm_tac o rewrite_rule[get_spec⌜Distinct⌝]));
a(lemma_tac⌜Elems list1 = Elems (list2 ↑ ∼ {x})⌝);
(* *** Goal "2.2.2" *** *)
a(rewrite_tac[↑_thm1]);
a(DROP_NTH_ASM_T 4 (rewrite_thm_tac o eq_sym_rule));
a(rewrite_tac[get_spec⌜Elems⌝]);
a(PC_T"hol1"(REPEAT strip_tac));
(* *** Goal "2.2.1.1" *** *)
a(PC_T"hol1"(rewrite_tac[]));
a(swap_asm_concl_tac⌜x′ ∈ Elems list1⌝ THEN asm_rewrite_tac[]);
(* *** Goal "2.2.1.2" *** *)
a(TOP_ASM_T (strip_asm_tac o rewrite_rule[]));
(* *** Goal "2.2.2" *** *)
a(DROP_NTH_ASM_T 6 (strip_asm_tac o ∀_elim⌜list2 ↑ ∼ {x}⌝));
a(rewrite_tac(map get_spec[⌜Length⌝]));
a(swap_nth_asm_concl_tac 5 THEN rewrite_tac(map get_spec[⌜$Append⌝, ⌜Length⌝]));
a(lemma_tac⌜x ∈ Elems list2⌝);
(* *** Goal "2.2.2.1" *** *)
a(GET_NTH_ASM_T 6 (fn th => rewrite_tac[eq_sym_rule th, get_spec⌜Elems⌝]));
```

```
a(PC_T"hol1" (REPEAT strip_tac));
(* *** Goal "2.2.2.2" *** *)
a(strip_asm_tac(list_∀_elim[⌜list2⌝, ⌜x⌝]↾_thm4));
a(swap_nth_asm_concl_tac 1);
a(rewrite_tac[get_spec⌜$<⌝]);
a(strip_asm_tac(list_∀_elim
        [⌜Length list1 + 1⌝, ⌜Length (list2 ↾ ~ {x}) + 1⌝, ⌜Length list2⌝]
        ≤_trans_thm));
pop_thm()
));
```

SML

```
val size_thm1 = save_thm("size_thm1", (
push_goal([], ⌜Size {} = 0⌝);
a(rewrite_tac[get_spec⌜Size⌝, elems_thm1]);
a(conv_tac (ONCE_MAP_C prove_∃_conv));
a(rewrite_tac[get_spec⌜Length⌝]);
a(lemma_tac⌜{i|0 = i} = {0}⌝ );
(* *** Goal "1" *** *)
a(PC_T"hol1" (REPEAT strip_tac)
        THEN TRY_T (asm_ante_tac ⌜¬x = 0⌝) THEN asm_rewrite_tac[]);
(* *** Goal "2" *** *)
a(asm_rewrite_tac[min_clauses]);
pop_thm()
));
```

SML

```
val size_thm2 = save_thm("size_thm2", (
push_goal([], ⌜∀list• list ∈ Distinct ⇒ Size(Elems list) = Length list⌝);
a(rewrite_tac[get_spec⌜Size⌝] THEN REPEAT strip_tac);
a(strip_asm_tac (list_∀_elim
        [⌜Length list⌝, ⌜{i|∃ list'• Length list' = i ∧ Elems list' = Elems list}⌝]
        min_thm));
(* *** Goal "1" *** *)
a(POP_ASM_T (strip_asm_tac o rewrite_rule[] o ∀_elim⌜list⌝));
a(TOP_ASM_T ante_tac THEN GET_NTH_ASM_T 3 (rewrite_thm_tac o eq_sym_rule));
a(strip_tac THEN strip_asm_tac (list_∀_elim
        [⌜Length list⌝, ⌜Length list'⌝]
        ≤_antisym_thm));
(* *** Goal "2.1" *** *)
a(strip_asm_tac (list_∀_elim[⌜list⌝, ⌜list'⌝]distinct_thm1));
a(asm_ante_tac ⌜¬ Elems list = Elems list'⌝);
a(GET_NTH_ASM_T 5 rewrite_thm_tac);
```

```
(* *** Goal "2.2" *** *)
a(GET_NTH_ASM_T 2 rewrite_thm_tac);
pop_thm()
));
```

3.5 Relationship between *Finite* and Set Operations

SML

```
val fin_set_thm2 = save_thm("fin_set_thm2", (
push_goal([], ⌜{} ∈ Finite⌝);
a(rewrite_tac[get_spec⌜Finite⌝]);
a(PC_T "hol1" (REPEAT strip_tac));
pop_thm()
));
```

SML

```
val fin_set_thm3 = save_thm("fin_set_thm3", (
push_goal([], ⌜∀ a x• a ∈ Finite ⇒ ({x} ∪ a) ∈ Finite⌝);
a(rewrite_tac[get_spec⌜Finite⌝] THEN (PC_T "hol1" (REPEAT strip_tac)));
a(spec_nth_asm_tac 3 ⌜s⌝);
(* *** Goal "1" *** *)
a(list_spec_nth_asm_tac 3 [⌜a'⌝, ⌜x'⌝]);
(* *** Goal "2" *** *)
a(list_spec_nth_asm_tac 2 [⌜a⌝, ⌜x⌝]);
pop_thm()
));
```

SML

```
val fin_set_thm4 = save_thm("fin_set_thm4", (
push_goal([], ⌜∀ list• Elems list ∈ Finite⌝);
a(strip_tac);
a(list_induction_tac⌜list⌝);
(* *** Goal "1" *** *)
a(rewrite_tac[get_spec ⌜Elems⌝, fin_set_thm2]);
(* *** Goal "2" *** *)
a(rewrite_tac[get_spec ⌜Elems⌝]);
a(strip_tac);
a(strip_asm_tac (list_∀_elim[⌜Elems list⌝, ⌜x⌝]fin_set_thm3));
pop_thm()
));
```

SML

```
val size_thm3 = save_thm("size_thm3", (
push_goal([], ⌜∀a b• a ∈ Finite ∧ b ∈ Finite
        ⇒                (a ∪ b) ∈ Finite⌝);
a(REPEAT strip_tac);
a(strip_asm_tac (∀_elim ⌜a⌝ fin_set_thm1));
a(strip_asm_tac (∀_elim ⌜b⌝ fin_set_thm1));
a(asm_rewrite_tac[conv_rule(ONCE_MAP_C eq_sym_conv) elems_thm3]);
a(prove_tac[fin_set_thm4]);
pop_thm()
));
```

SML

```
val size_thm4 = save_thm("size_thm4", (
push_goal([], ⌜∀a b• a ∈ Finite ∧ b ⊆ a
        ⇒                b ∈ Finite⌝);
a(REPEAT strip_tac);
a(strip_asm_tac(∀_elim⌜a⌝fin_set_thm1));
a(LEMMA_T ⌜b = Elems (list ↑ b)⌝ once_rewrite_thm_tac);
(* *** Goal "1" *** *)
a(DROP_NTH_ASM_T 2 (fn th => rewrite_tac[eq_sym_rule th, ↑_thm1]));
a(PC_T"hol1"(asm_prove_tac[]));
(* *** Goal "2" *** *)
a(rewrite_tac[fin_set_thm4]);
pop_thm()
));
```

SML

```
val size_thm5 = save_thm("size_thm5", (
push_goal([], ⌜∀a x• a ∈ Finite
        ⇒        Size({x} ∪ a) = if x ∈ a then Size a else Size a + 1⌝);
a(REPEAT strip_tac);
a(cases_tac⌜x ∈ a⌝);
(* *** Goal "1" *** *)
a(LEMMA_T ⌜{x} ∪ a = a⌝ asm_rewrite_thm_tac);
a(PC_T"hol1"(REPEAT strip_tac));
a(asm_rewrite_tac[]);
(* *** Goal "2" *** *)
a(strip_asm_tac(list_∀_elim[⌜a⌝]fin_set_thm1));
a(lemma_tac⌜{x} ∪ a = Elems (Cons x list) ∧ Cons x list ∈ Distinct⌝);
(* *** Goal "2.1" *** *)
a(asm_rewrite_tac[get_spec⌜Elems⌝, get_spec⌜Distinct⌝]);
a(DROP_NTH_ASM_T 2 (asm_rewrite_thm_tac o eq_sym_rule));
```

```
(* *** Goal "2.2" *** *)
a(strip_asm_tac(list_∀_elim[⌜list⌝]size_thm2));
a(strip_asm_tac(list_∀_elim[⌜Cons x list⌝]size_thm2));
a(LIST_GET_NTH_ASM_T [7, 4, 1] rewrite_tac);
a(LIST_GET_NTH_ASM_T [6, 2] rewrite_tac);
a(rewrite_tac[get_spec⌜Length⌝]);
pop_thm()
));
```

SML

```
val size_thm6 = save_thm("size_thm6", (
push_goal([], ⌜∀a b• a ∈ Finite ∧ b ∈ Finite ⇒ (a ∩ b) ∈ Finite⌝);
a(REPEAT strip_tac);
a(strip_asm_tac(pc_rule"hol1"(prove_rule[])⌜a ∩ b ⊆ a⌝));
a(strip_asm_tac(list_∀_elim[⌜a⌝, ⌜a ∩ b⌝]size_thm4));
pop_thm()
));
```

SML

```
val size_thm7 = save_thm("size_thm7", (
push_goal([], ⌜∀a b• a ∈ Finite ∧ b ∈ Finite
        ⇒              Size(a ∪ b) + Size (a ∩ b) = Size a + Size b⌝);
a(REPEAT strip_tac);
a(asm_ante_tac ⌜a ∈ Finite⌝ THEN fin_set_induction_tac);
(* *** Goal "1" *** *)
a(rewrite_tac[size_thm1]);
(* *** Goal "2" *** *)
a(strip_asm_tac(list_∀_elim[⌜a⌝, ⌜b⌝]size_thm3));
a(strip_asm_tac(list_∀_elim[⌜a⌝, ⌜b⌝]size_thm6));
a(cases_tac⌜x ∈ b⌝);
(* *** Goal "2.1" *** *)
a(lemma_tac ⌜({x} ∪ a) ∪ b = a ∪ b ∧
        ({x} ∪ a) ∩ b = {x} ∪ (a ∩ b)⌝);
(* *** Goal "2.1.1" *** *)
a(PC_T"hol1"(REPEAT strip_tac THEN asm_rewrite_tac[]));
(* *** Goal "2.1.2" *** *)
a(strip_asm_tac(list_∀_elim[⌜a ∪ b⌝, ⌜x⌝]size_thm5));
a(strip_asm_tac(list_∀_elim[⌜a ∩ b⌝, ⌜x⌝]size_thm5));
a(strip_asm_tac(list_∀_elim[⌜a⌝, ⌜x⌝]size_thm5));
a(LEMMA_T ⌜x ∈ a ∪ b ∧ ¬x ∈ a ∩ b⌝
        (fn th => asm_rewrite_tac[th, ∀_elim⌜1⌝plus_order_thm]));
a(REPEAT strip_tac);
(* *** Goal "2.2" *** *)
```

```
a(lemma_tac ⌜({x} ∪ a) ∪ b = {x} ∪ a ∪ b ∧
        ({x} ∪ a) ∩ b = (a ∩ b)⌝);
(* *** Goal "2.2.1" *** *)
a(PC_T"hol1"(REPEAT strip_tac THEN asm_rewrite_tac[]));
a(asm_ante_tac ⌜x' ∈ b⌝ THEN asm_rewrite_tac[]);
(* *** Goal "2.2.2" *** *)
a(strip_asm_tac(list_∀_elim[⌜a ∪ b⌝, ⌜x⌝]size_thm5));
a(strip_asm_tac(list_∀_elim[⌜a ∩ b⌝, ⌜x⌝]size_thm5));
a(strip_asm_tac(list_∀_elim[⌜a⌝, ⌜x⌝]size_thm5));
a(LEMMA_T ⌜¬x ∈ a ∪ b ∧ ¬x ∈ a ∩ b⌝
        (fn th => asm_rewrite_tac[th, ∀_elim⌜1⌝plus_order_thm]));
a(REPEAT strip_tac);
pop_thm()
));
```

SML

```
val size_singleton_thm = save_thm("size_singleton_thm", (
push_goal([], ⌜∀x• #{x} = 1⌝);
a strip_tac;
a(accept_tac(rewrite_rule[fin_set_thm2, size_thm1]
        (list_∀_elim[⌜{}⌝, ⌜x⌝]size_thm5)));
pop_thm()
));
```

3.6 Miscellany

SML

```
val ℕ_set_thm1 = save_thm("ℕ_set_thm1", (
push_goal([], ⌜∀a: ℕ SET• a ∈ Finite ∧ ¬a = {} ⇒ ∃m•m ∈ a ∧ ∀i•i ∈ a ⇒ i ≤ m⌝);
a(strip_tac THEN bc_tac[taut_rule⌜(A ⇒ B ⇒ C) ⇒ (A ∧ B ⇒ C)⌝]);
a(fin_set_induction_tac THEN_TRY asm_rewrite_tac[] THEN REPEAT strip_tac);
(* *** Goal "1" *** *)
a(∃_tac⌜x:ℕ⌝ THEN REPEAT strip_tac THEN asm_rewrite_tac[]);
(* *** Goal "2" *** *)
a(strip_asm_tac(list_∀_elim[⌜x⌝, ⌜m⌝]≤_cases_thm));
(* *** Goal "2.1" *** *)
a(∃_tac⌜m⌝ THEN REPEAT strip_tac THEN_TRY asm_rewrite_tac[]);
a(asm_fc_tac[]);
(* *** Goal "2.2" *** *)
a(∃_tac⌜x⌝ THEN REPEAT strip_tac THEN_TRY asm_rewrite_tac[]);
a(asm_fc_tac[] THEN rename_tac[(⌜i⌝, "ii")]);
a(bc_tac[≤_trans_thm]);
a(∃_tac⌜m⌝ THEN REPEAT strip_tac);
```

```
pop_thm()
));
```

SML

```
val finite_max_thm = save_thm("finite_max_thm", (
push_goal([], ⌜∀a: ℕ SET• a ∈ Finite ∧ ¬a = {} ⇒ Max a ∈ a ∧ ∀i•i ∈ a ⇒ i ≤ Max a⌝);
a(REPEAT_N 2 strip_tac);
a(fc_tac[ℕ_set_thm1]);
a(LEMMA_T ⌜Max a = m⌝ (fn th => rewrite_tac[th] THEN REPEAT strip_tac));
(* *** Goal "1" *** *)
a(bc_tac[get_spec⌜Max⌝] THEN REPEAT strip_tac THEN asm_fc_tac[]);
(* *** Goal "2" *** *)
a(asm_fc_tac[]);
pop_thm()
));
```

SML

```
val finite_size_thm = save_thm("finite_size_thm", (
push_goal([], ⌜∀a m•
        (∃list•Elems list = a ∧ list ∈ Distinct ∧ Length list = m)
⇔       a ∈ Finite ∧ #a = m
⌝);
a(REPEAT strip_tac);
(* *** Goal "1" *** *)
a(GET_NTH_ASM_T 3 (fn th => rewrite_tac[eq_sym_rule th, fin_set_thm4]));
(* *** Goal "2" *** *)
a(fc_tac[size_thm2]);
a(POP_ASM_T ante_tac THEN asm_rewrite_tac[]);
(* *** Goal "3" *** *)
a(fc_tac[fin_set_thm1]);
a(∃_tac⌜list⌝ THEN asm_rewrite_tac[]);
a(fc_tac[size_thm2]);
a(LIST_DROP_NTH_ASM_T [1,3,4] (rewrite_tac o map eq_sym_rule));
pop_thm()
));
```

SML

```
val length_map_thm = save_thm("length_map_thm", (
push_goal([], ⌜∀f list•
        Length(Map f list) = Length list
⌝);
a(REPEAT strip_tac);
a(list_induction_tac⌜list⌝ THEN asm_rewrite_tac[length_def, map_def]);
```

```
|pop_thm()
|));
```

SML

```
|val elems_map_thm = save_thm("elems_map_thm", (
|push_goal([], ⌜∀f list•
|        Elems(Map f list) = {y | ∃x• x ∈ Elems list ∧ f x = y}
|⌝);
|a(REPEAT strip_tac);
|a(list_induction_tac⌜list⌝ THEN asm_rewrite_tac[elems_def, map_def]
|        THEN PC_T"hol1"(REPEAT strip_tac));
|(* *** Goal "1" *** *)
|a(∃_tac⌜x⌝ THEN PC_T"hol1"(REPEAT strip_tac) THEN asm_rewrite_tac[]);
|(* *** Goal "2" *** *)
|a(∃_tac⌜x''⌝ THEN PC_T"hol1"(REPEAT strip_tac));
|(* *** Goal "3" *** *)
|a(POP_ASM_T ante_tac THEN POP_ASM_T ante_tac THEN asm_rewrite_tac[]);
|a(strip_tac THEN asm_rewrite_tac[]);
|(* *** Goal "4" *** *)
|a(∃_tac⌜x''⌝ THEN PC_T"hol1"(REPEAT strip_tac));
|pop_thm()
|));
```

SML

```
|val map_distinct_thm = save_thm("map_distinct_thm", (
|push_goal([], ⌜∀f list•
|        (Map f list) ∈ Distinct
|⇔       list ∈ Distinct
|        ∧ (∀x y• x ∈ Elems list ∧ y ∈ Elems list ∧ f x = f y ⇒ x = y)
|⌝);
|a(REPEAT ∀_tac);
|a(list_induction_tac⌜list⌝);
|(* *** Goal "1" of 5 *** *)
|a(asm_rewrite_tac[elems_def, map_def, distinct_def]);
|(* *** Goal "2" *** *)
|a(asm_rewrite_tac[elems_def, map_def, distinct_def]);
|(* *** Goal "3" *** *)
|a(rename_tac[]);
|a(asm_rewrite_tac[elems_def, map_def, distinct_def]);
|a(PC_T"hol1"(REPEAT strip_tac));
|a(asm_fc_tac[] THEN asm_fc_tac[]);
|(* *** Goal "4" *** *)
|a(asm_fc_tac[] THEN asm_fc_tac[]);
```

```
(* *** Goal "5" *** *)
a(asm_rewrite_tac[elems_def, map_def, distinct_def]);
a(rewrite_tac[taut_rule⌜∀p q r•(¬p ⇔ ¬q ∧ r) ⇔ (p ⇔ q ∨ ¬r)⌝, elems_map_thm]);
a(REPEAT_N 3 strip_tac);
(* *** Goal "5.1" *** *)
a(strip_tac THEN cases_tac ⌜x ∈ Elems list⌝ THEN1 REPEAT strip_tac);
a(PC_T"hol1"(REPEAT strip_tac));
a(list_spec_nth_asm_tac 1 [⌜x'⌝, ⌜x⌝]);
a(POP_ASM_T (rewrite_thm_tac o eq_sym_rule) THEN strip_tac);
(* *** Goal "5.2" *** *)
a(cases_tac ⌜x ∈ Elems list⌝);
(* *** Goal "5.2.1" *** *)
a(⇒_T (fn th => id_tac));
a(∃_tac⌜x⌝ THEN asm_rewrite_tac[]);
(* *** Goal "5.2.2" *** *)
a(asm_rewrite_tac[]);
a(once_rewrite_tac[taut_rule⌜∀p q•(¬p ⇒ q) ⇔ (¬q ⇒ p)⌝]);
a(PC_T"hol1"(REPEAT strip_tac));
(* *** Goal "5.2.2.1" *** *)
a(asm_rewrite_tac[]);
(* *** Goal "5.2.2.2" *** *)
a(list_spec_nth_asm_tac 4 [⌜y⌝]);
a(POP_ASM_T ante_tac THEN POP_ASM_T ante_tac THEN asm_rewrite_tac[]);
a(⇒_T rewrite_thm_tac);
(* *** Goal "5.2.2.3" *** *)
a(list_spec_nth_asm_tac 4 [⌜x'⌝]);
a(POP_ASM_T ante_tac THEN POP_ASM_T ante_tac THEN asm_rewrite_tac[]);
a(⇒_T rewrite_thm_tac);
(* *** Goal "5.2.2.4" *** *)
a(asm_fc_tac[] THEN asm_fc_tac[]);
pop_thm()
));
```

SML

```
val pair_eq_thm = save_thm("pair_eq_thm", (
push_goal([], ⌜∀x y• Fst x = Fst y ∧ Snd x = Snd y ⇒ x = y⌝);
a(REPEAT strip_tac);
a(LEMMA_T⌜y = (Fst x, Snd x)⌝ rewrite_thm_tac);
a(pure_asm_rewrite_tac[] THEN rewrite_tac[]);
pop_thm()
));
```

SML

```
val at_thm = save_thm("at_thm", (
push_goal([], ⌜∀f: 'a ↔ 'b; x : 'a•
        f ∈ Functional ∧ x ∈ Dom f
        ⇒ ∀y•f@x = y ⇔ (x, y) ∈ f
⌝);
a(rewrite_tac[functional_def, dom_def]);
a(REPEAT strip_tac);
(* *** Goal "1" *** *)
a(fc_tac[get_spec⌜$At⌝]);
(* *** Goal "1.1" *** *)
a(asm_fc_tac[] THEN asm_fc_tac[]);
a(asm_ante_tac⌜¬ z = y⌝ THEN asm_rewrite_tac[]);
(* *** Goal "1.2" *** *)
a(DROP_ASM_T⌜f @ x = y'⌝(asm_rewrite_thm_tac o eq_sym_rule));
(* *** Goal "2" *** *)
a(fc_tac[get_spec⌜$At⌝]);
(* *** Goal "2.1" *** *)
a(asm_fc_tac[] THEN asm_fc_tac[]);
(* *** Goal "2.2" *** *)
a(asm_rewrite_tac[] THEN asm_fc_tac[] THEN asm_fc_tac[]);
pop_thm()
));
```

SML

```
val finite_function_thm = save_thm("finite_function_thm", (
push_goal([], ⌜∀f: 'a ↔ 'b•
                f ∈ Functional
        ∧       (f ∈ Finite ∨ Dom f ∈ Finite)
        ⇒       f ∈ Finite ∧ Dom f ∈ Finite ∧ #(Dom f) = #f
⌝);
a(REPEAT_N 2 strip_tac);
(* *** Goal "1" *** *)
a(asm_rewrite_tac[]);
a(fc_tac[fin_set_thm1]);
a(bc_tac[finite_size_thm]);
a(∃_tac⌜Map Fst list⌝);
a(fc_tac[size_thm2]);
a(asm_rewrite_tac[length_map_thm, elems_map_thm, map_distinct_thm, dom_def]);
a(PC_T"hol1"(REPEAT strip_tac));
(* *** Goal "1.1" *** *)
a(∃_tac⌜Snd x'⌝ THEN POP_ASM_T (asm_rewrite_thm_tac o eq_sym_rule));
(* *** Goal "1.2" *** *)
```

```
a(∃_tac⌜(x, y)⌝ THEN asm_rewrite_tac[]);
(* *** Goal "1.3" *** *)
a(DROP_ASM_T⌜f = Elems list⌝ (asm_tac o eq_sym_rule));
a(LIST_DROP_NTH_ASM_T [3,4] (MAP_EVERY ante_tac) THEN asm_rewrite_tac[]);
a(DROP_ASM_T⌜f ∈ Functional⌝ (strip_asm_tac o rewrite_rule[get_spec⌜Functional⌝]));
a(REPEAT strip_tac);
a(lemma_tac ⌜(Fst x, Snd y) ∈ f⌝ THEN1 asm_rewrite_tac[]);
a(LIST_SPEC_NTH_ASM_T 4 [⌜Fst x⌝, ⌜Snd x⌝, ⌜Snd y⌝] (strip_asm_tac o rewrite_rule[]));
a(bc_tac[pair_eq_thm] THEN REPEAT strip_tac);
(* *** Goal "2" *** *)
a(asm_rewrite_tac[]);
a(fc_tac[fin_set_thm1]);
a(conv_tac(ONCE_MAP_C eq_sym_conv) THEN bc_tac[finite_size_thm]);
a(∃_tac⌜Map (λx•(x, f@x)) list⌝);
a(fc_tac[size_thm2]);
a(asm_rewrite_tac[length_map_thm, elems_map_thm, map_distinct_thm, dom_def]);
a(PC_T"hol1"(REPEAT strip_tac));
(* *** Goal "2.1" *** *)
a(DROP_NTH_ASM_T 6 (asm_tac o eq_sym_rule));
a(DROP_NTH_ASM_T 4 ante_tac THEN asm_rewrite_tac[] THEN strip_tac);
a(fc_tac[at_thm] THEN asm_fc_tac[]);
a(DROP_NTH_ASM_T 5 ante_tac THEN asm_rewrite_tac[] THEN strip_tac);
a(asm_fc_tac[]);
a(POP_ASM_T ante_tac THEN rewrite_tac[]);
(* *** Goal "2.2" *** *)
a(lemma_tac⌜Fst x ∈ Dom f⌝);
(* *** Goal "2.2.1" *** *)
a(rewrite_tac[dom_def] THEN ∃_tac⌜Snd x⌝ THEN asm_rewrite_tac[]);
(* *** Goal "2.2.2" *** *)
a(lemma_tac⌜x = (Fst x, f@(Fst x))⌝);
(* *** Goal "2.2.2.1" *** *)
a(fc_tac[conv_rule (ONCE_MAP_C eq_sym_conv) at_thm] THEN asm_fc_tac[]);
a(POP_ASM_T (ante_tac o ∀_elim⌜Snd x⌝) THEN rewrite_tac[]);
a(strip_tac THEN asm_rewrite_tac[]);
(* *** Goal "2.2.2.2" *** *)
a(DROP_NTH_ASM_T 6 (rewrite_thm_tac o eq_sym_rule));
a(POP_ASM_T once_rewrite_thm_tac THEN ∃_tac⌜Fst x⌝ THEN asm_rewrite_tac[]);
pop_thm()
));
```

4 THE THEORY fin_thms

4.1 Parents

fin_set

4.2 Theorems

fin_set_induction_thm
$\vdash \forall\ P$
$\bullet\ P\ \{\}$
$\wedge\ (\forall\ a\ x$
$\bullet\ a \in Finite \wedge P\ a \wedge \neg\ x \in a \Rightarrow P\ (\{x\} \cup a))$
$\Rightarrow\ (\forall\ a \bullet\ a \in Finite \Rightarrow P\ a)$

min_thm $\vdash \forall\ m\ a \bullet\ m \in a \Rightarrow Min\ a \in a \wedge Min\ a \leq m$

min_clauses $\vdash (\forall\ m \bullet\ Min\ \{m\} = m)$
$\wedge\ (\forall\ m\ n \bullet\ Min\ \{m;\ n\} = (if\ m \leq n\ then\ m\ else\ n))$
$\wedge\ (\forall\ m\ a$
$\bullet\ Min\ (\{m\} \cup a)$
$= (if\ a = \{\}\ then\ m\ else\ Min\ \{m;\ Min\ a\}))$

fin_set_thm1 $\vdash \forall\ a$
$\bullet\ a \in Finite$
$\Rightarrow\ (\exists\ list \bullet\ a = Elems\ list \wedge list \in Distinct)$

elems_thm1 $\vdash \forall\ list \bullet\ Elems\ list = \{\} \Leftrightarrow list = []$

elems_thm2 $\vdash (\forall\ list \bullet\ Elems\ list = \{\} \Leftrightarrow list = [])$
$\wedge\ (\forall\ list \bullet\ \{\} = Elems\ list \Leftrightarrow list = [])$

elems_thm3 $\vdash \forall\ list1\ list2$
$\bullet\ Elems\ (list1 \frown list2) = Elems\ list1 \cup Elems\ list2$

length_thm $\vdash \forall\ list1\ list2 \bullet\ \#\ (list1 \frown list2) = \#\ list1 + \#\ list2$

↾_thm1 $\vdash \forall\ list\ a \bullet\ Elems\ (list \upharpoonright a) = Elems\ list \cap a$

↾_thm2 $\vdash \forall\ list\ a \bullet\ \#\ ((list \upharpoonright a) \frown (list \upharpoonright \sim a)) = \#\ list$

↾_thm3 $\vdash \forall\ list\ a \bullet\ Elems\ list \subseteq a \Rightarrow list \upharpoonright a = list$

↾_thm4 $\vdash \forall\ list\ x \bullet\ x \in Elems\ list \Rightarrow \#\ (list \upharpoonright \sim \{x\}) < \#\ list$

distinct_thm1
$\vdash \forall\ list1\ list2$
$\bullet\ list1 \in Distinct \wedge Elems\ list1 = Elems\ list2$
$\Rightarrow\ \#\ list1 \leq \#\ list2$

size_thm1 $\vdash \#\ \{\} = 0$

size_thm2 $\vdash \forall\ list \bullet\ list \in Distinct \Rightarrow \#\ (Elems\ list) = \#\ list$

fin_set_thm2 $\vdash \{\} \in Finite$

fin_set_thm3 $\vdash \forall\ a\ x \bullet\ a \in Finite \Rightarrow \{x\} \cup a \in Finite$

fin_set_thm4 $\vdash \forall\ list \bullet\ Elems\ list \in Finite$

size_thm3 $\vdash \forall\ a\ b \bullet\ a \in Finite \wedge b \in Finite \Rightarrow a \cup b \in Finite$

size_thm4 $\vdash \forall\ a\ b \bullet\ a \in Finite \wedge b \subseteq a \Rightarrow b \in Finite$

size_thm5 $\vdash \forall\ a\ x$
$\bullet\ a \in Finite$
$\Rightarrow\ \#\ (\{x\} \cup a) = (if\ x \in a\ then\ \#\ a\ else\ \#\ a + 1)$

size_thm6 $\vdash \forall\ a\ b \bullet\ a \in Finite \wedge b \in Finite \Rightarrow a \cap b \in Finite$

size_thm7 $\vdash \forall\ a\ b$
$\bullet\ a \in Finite \wedge b \in Finite$
$\Rightarrow \#\ (a \cup b) + \#\ (a \cap b) = \#\ a + \#\ b$

size_singleton_thm
$\vdash \forall\ x \bullet\ \#\ \{x\} = 1$

$\mathbb{N}$***_set_thm1*** $\vdash \forall\ a$
$\bullet\ a \in Finite \wedge \neg\ a = \{\}$
$\Rightarrow (\exists\ m \bullet\ m \in a \wedge (\forall\ i \bullet\ i \in a \Rightarrow i \leq m))$

finite_max_thm
$\vdash \forall\ a$
$\bullet\ a \in Finite \wedge \neg\ a = \{\}$
$\Rightarrow Max\ a \in a \wedge (\forall\ i \bullet\ i \in a \Rightarrow i \leq Max\ a)$

finite_size_thm
$\vdash \forall\ a\ m$
$\bullet\ (\exists\ list$
$\bullet\ Elems\ list = a \wedge list \in Distinct \wedge \#\ list = m)$
$\Leftrightarrow a \in Finite \wedge \#\ a = m$

length_map_thm
$\vdash \forall\ f\ list \bullet\ \#\ (Map\ f\ list) = \#\ list$

elems_map_thm
$\vdash \forall\ f\ list$
$\bullet\ Elems\ (Map\ f\ list)$
$= \{y | \exists\ x \bullet\ x \in Elems\ list \wedge f\ x = y\}$

map_distinct_thm
$\vdash \forall\ f\ list$
$\bullet\ Map\ f\ list \in Distinct$
$\Leftrightarrow list \in Distinct$
$\wedge (\forall\ x\ y$
$\bullet\ x \in Elems\ list \wedge y \in Elems\ list \wedge f\ x = f\ y$
$\Rightarrow x = y)$

pair_eq_thm $\vdash \forall\ x\ y \bullet\ Fst\ x = Fst\ y \wedge Snd\ x = Snd\ y \Rightarrow x = y$

at_thm $\vdash \forall\ f\ x$
$\bullet\ f \in Functional \wedge x \in Dom\ f$
$\Rightarrow (\forall\ y \bullet\ f\ @\ x = y \Leftrightarrow (x, y) \in f)$

finite_function_thm
$\vdash \forall\ f$
$\bullet\ f \in Functional \wedge (f \in Finite \vee Dom\ f \in Finite)$
$\Rightarrow f \in Finite \wedge Dom\ f \in Finite \wedge \#\ (Dom\ f) = \#\ f$

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