

Mathematical Case Studies: the Complex Numbers*

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30 December 2016

Abstract

Definitions of the complex numbers and their arithmetic operators in **ProofPower-HOL** with proofs of some of their elementary algebraic properties.

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Reference: LEMMA1/HOL/WRK072; Current git revision: b0522be

*First posted 17 September 2005; for full changes history see: <https://github.com/RobArthan/pp-contrib>.

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1 INTRODUCTION

This document contains the beginnings of a theory of complex arithmetic in ProofPower-HOL.

After some preliminaries, section 2 introduces the field operations and the embeddings of the real numbers and of conjugation (which help to abbreviate the definition of reciprocals). We then define the derived operations of binary subtraction and division and the embedding of the natural numbers.

In section 4 we introduce the topology on the complex numbers and define the exponential mapping of the real line onto the unit circle.

A proof script contained in the source of this document but hidden from the printed document proves several theorems about the complex numbers. A listing of the theorems proved is given in appendix A.

We prove that the complex numbers form a field and prove some elementary algebraic properties of complex conjugation and of the embeddings of the natural numbers and the real numbers in the complex numbers. The proofs of these basic algebraic properties generally comprise little more than simplifying using definitions and standard theorems and then application of the decision procedure for linear arithmetic to prove the resulting problem of real algebra. One exception is proving that $zz^{-1} = 1$, which requires a few lines of reasoning about real squares.

The proof scripts then go on to develop basic facts about the exponential mapping of the real line onto the unit circle, leading up to the important fact that it is a covering projection.

2 THE ALGEBRA \mathbb{C}

2.1 Preliminaries

The following commands set up a theory to hold the definitions, theorems, etc. The theory has the theory of analysis defined in [1] for its parent, although the elementary algebraic theory only needs one or two little theorems about squares.

```
SML
|force_delete_theory"C" handle Fail _ => ();
|open_theory"analysis";
|new_theory"C";
|new_parent"topology_ℝ";
```

Now set up a convenient proof context:

```
SML
|set_merge_pcs["basic_hol1", "'ℤ", "'ℝ", "'sets_alg"];
```

2.2 The Type \mathbb{C}

The type \mathbb{C} comprises pairs of real numbers. We capture this in a type abbreviation:

```
SML
|declare_type_abbrev("C", [], [ℝ × ℝ]);
```

We declare aliases for the instances of the projection functions that give the real and imaginary part of a complex number:

```
SML
| app declare_alias [
|   ("Re", 「Fst : ℂ → ℝ」),
|   ("Im", 「Snd : ℂ → ℝ」)];
```

2.3 Algebraic Operators

2.3.1 Addition

```
SML
| declare_infix(300, "+_C");
```

```
HOL Constant
| $+_C : ℂ → ℂ → ℂ
|-----
| ∀ z w : ℂ • z +_C w = (Re z + Re w, Im z + Im w)
```

```
SML
| declare_alias("+", 「$+_C」);
```

2.3.2 Negation

```
HOL Constant
| ~_C : ℂ → ℂ
|-----
| ∀ z : ℂ • ~_C z = (~(Re z), ~(Im z))
```

```
SML
| declare_alias("~", 「~_C」);
```

2.3.3 Multiplication

```
SML
| declare_infix(310, "*_C");
```

```
HOL Constant
| $*_C : ℂ → ℂ → ℂ
|-----
| ∀ z w : ℂ •
|   z *_C w = (Re z * Re w - Im z * Im w, Re z * Im w + Im z * Re w)
```

```
SML
| declare_alias("*", 「$*_C」);
```

2.3.4 Embedding of the Real Numbers

HOL Constant

| $\mathbb{R}\mathbb{C} : \mathbb{R} \rightarrow \mathbb{C}$

| $\forall x \bullet \mathbb{R}\mathbb{C} x = (x, \text{NR } 0)$

HOL Constant

| $\mathbb{R}i : \mathbb{R} \rightarrow \mathbb{C}$

| $\forall x \bullet \mathbb{R}i x = (\text{NR } 0, x)$

2.3.5 The Imaginary Unit

HOL Constant

| $I_C : \mathbb{C}$

| $I_C = \mathbb{R}i 1.0$

2.3.6 Conjugation

SML

| `declare_postfix(320, "-");`

HOL Constant

| $\mathbb{S}^- : \mathbb{C} \rightarrow \mathbb{C}$

| $\forall z : \mathbb{C} \bullet z^- = (\text{Re } z, \sim(\text{Im } z))$

2.3.7 Reciprocal

SML

| `declare_postfix(320, "-1C");`

HOL Constant

| $\mathbb{S}^{-1C} : \mathbb{C} \rightarrow \mathbb{C}$

| $\forall z : \mathbb{C} \bullet z^{-1C} = z^- * \mathbb{R}\mathbb{C}(\text{Re } (z * z^-)^{-1})$

SML

| `declare_alias("-1", "⌈ \mathbb{S}^{-1C} ⌋");`

2.3.8 Subtraction

SML

| `declare_infix(300, "-C");`

HOL Constant

| $\$-_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C} \rightarrow \mathbb{C}$

| $\forall z w : \mathbb{C} \bullet z -_{\mathbb{C}} w = z + \sim w$

2.3.9 Division

SML

| `declare_infix(315, "/_C");`

HOL Constant

| $\$/_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C} \rightarrow \mathbb{C}$

| $\forall z w : \mathbb{C} \bullet z /_{\mathbb{C}} w = z * (w^{-1})$

SML

| `declare_alias("/_", "\$/_C\^{\top}");`

2.3.10 Embedding of the Natural Numbers

HOL Constant

| $\mathbb{N}_{\mathbb{C}} : \mathbb{N} \rightarrow \mathbb{C}$

| $\forall m \bullet \mathbb{N}_{\mathbb{C}} m = \mathbb{R}_{\mathbb{C}}(\mathbb{N}_{\mathbb{R}} m)$

HOL Constant

| $\mathbb{N}_{\mathbb{C}} : \mathbb{N} \rightarrow \mathbb{C}$

| $\forall m \bullet \mathbb{N}_{\mathbb{C}} m = (0., \mathbb{N}_{\mathbb{R}} m)$

2.3.11 Exponentiation with Natural Number Exponents

SML

| `declare_infix(320, "^_C");`

HOL Constant

| $\$^{\wedge}_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{N} \rightarrow \mathbb{C}$

| $(\forall z : \mathbb{C} \bullet z^{\wedge}_{\mathbb{C}} 0 = \mathbb{N}_{\mathbb{C}} 1)$
| $\wedge (\forall z : \mathbb{C}; m \bullet z^{\wedge}_{\mathbb{C}} (m+1) = z * z^{\wedge}_{\mathbb{C}} m)$

SML

| `declare_alias("^_", "\$^{\wedge}_{\mathbb{C}}\^{\top}");`

2.4 \mathbb{C} qua Real Normed Space

SML

```
| declare_infix(310, "*_RC");
```

HOL Constant

```
| \$*_RC :  $\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{C}$ 
```

```
|  $\forall x : \mathbb{R}; v : \mathbb{C} \bullet$ 
```

$$x *_RC v = \mathbb{R}\mathbb{C} x *_C v$$

HOL Constant

```
| Abs_C :  $\mathbb{C} \rightarrow \mathbb{R}$ 
```

```
|  $\forall v : \mathbb{C} \bullet$ 
```

$$Abs_C v = Sqrt(Re (v * v^{-}))$$

2.5 Transcendental Functions

2.5.1 Exponential

HOL Constant

```
| Exp_C :  $\mathbb{C} \rightarrow \mathbb{C}$ 
```

```
|  $\forall z \bullet Exp_C z = \mathbb{R}\mathbb{C} (Exp(Re z)) * (Cos (Im z), Sin (Im z))$ 
```

SML

```
| declare_alias("Exp", "\$Exp_C");
```

3 POLYNOMIALS

As we did for the real numbers in [1], we define the set of polynomial function on the complex numbers to be the smallest set of functions that contains all constant functions and the identity function and that is closed under pointwise addition and multiplication of functions.

HOL Constant

```
| PolyFunc_C :  $(\mathbb{C} \rightarrow \mathbb{C}) SET$ 
```

```
|  $PolyFunc_C = \bigcap$ 
```

```
| {  $A$ 
```

```
| |  $(\forall c \bullet (\lambda x \bullet c) \in A)$ 
```

```
|  $\wedge (\lambda x \bullet x) \in A$ 
```

```
|  $\wedge (\forall f g \bullet f \in A \wedge g \in A \Rightarrow (\lambda x \bullet f x + g x) \in A)$ 
```

```
|  $\wedge (\forall f g \bullet f \in A \wedge g \in A \Rightarrow (\lambda x \bullet f x * g x) \in A) \}$ 
```

The following function gives the n -partial sum of a series.

HOL Constant

Series_C : $(\mathbb{N} \rightarrow \mathbb{C}) \rightarrow (\mathbb{N} \rightarrow \mathbb{C})$

$(\forall s \bullet \text{Series}_C s \ 0 = \mathbb{N}C \ 0)$
 $\wedge (\forall s \ n \bullet \text{Series}_C s \ (n+1) = \text{Series}_C s \ n + s \ n)$

We represent a complex polynomial as a pair (s, n) where s is a sequence of coefficients and n is a bound on the degree. The following function maps such a pair to the polynomial function it represents.

HOL Constant

PolyEval_C : $(\mathbb{N} \rightarrow \mathbb{C}) \times \mathbb{N} \rightarrow \mathbb{C} \rightarrow \mathbb{C}$

$\forall s \ n \ z \bullet \text{PolyEval}_C (s, n) \ z = \text{Series}_C (\lambda i \bullet s \ i * z^i) \ (n+1)$

We now give the operations on coefficients that correspond to addition of polynomial functions ...

HOL Constant

PlusCoeffs_C : $((\mathbb{N} \rightarrow \mathbb{C}) \times \mathbb{N}) \rightarrow ((\mathbb{N} \rightarrow \mathbb{C}) \times \mathbb{N}) \rightarrow ((\mathbb{N} \rightarrow \mathbb{C}) \times \mathbb{N})$

$\forall s \ m \ t \ n \bullet$
 $\text{PlusCoeffs}_C (s, m) (t, n) =$
 $((\lambda i \bullet (\text{if } i \leq m \text{ then } s \ i \text{ else } \mathbb{N}C \ 0) +$
 $(\text{if } i \leq n \text{ then } t \ i \text{ else } \mathbb{N}C \ 0)), m+n)$

... and to multiplication of one polynomial function by another.

HOL Constant

TimesCoeffs_C : $((\mathbb{N} \rightarrow \mathbb{C}) \times \mathbb{N}) \rightarrow ((\mathbb{N} \rightarrow \mathbb{C}) \times \mathbb{N}) \rightarrow ((\mathbb{N} \rightarrow \mathbb{C}) \times \mathbb{N})$

$(\forall s \ t \ n \bullet \text{TimesCoeffs}_C (s, 0) (t, n) = ((\lambda i \bullet s \ 0 * t \ i), n))$
 $\wedge (\forall s \ m \ t \ n \bullet$
 $\text{TimesCoeffs}_C (s, m+1) (t, n) =$
 PlusCoeffs_C
 $(\text{TimesCoeffs}_C (s, m) (t, n))$
 $((\lambda i \bullet \text{if } i \leq m \text{ then } \mathbb{N}C \ 0 \text{ else } s \ (m+1) * t \ (i-(m+1))), m+n+1))$

4 TOPOLOGICAL ASPECTS

The standard topology on \mathbb{C} is just the product topology:

HOL Constant

Open_C : $\mathbb{C} \text{ SET SET}$

$\text{Open}_C = O_R \times_T O_R$

As with $Open_R$ it is convenient to have a short name for the topology:

SML

```
| declare_alias("O_C", "Open_C");
```

The unit circle:

HOL Constant

```
| S1 : C SET
```

```
| S1 = {z | Abs_C z = 1.}
```

and the exponential mapping of the real line onto the unit circle:

HOL Constant

```
| ExpS1 : R -> C
```

```
| forall t. ExpS1 t = (Cos t, Sin t)
```

References

- [1] LEMMA1/HOL/WRK066. *Mathematical Case Studies: Basic Analysis*. R.D. Arthan, Lemma 1 Ltd., rda@lemma-one.com.

A THEOREMS IN THE THEORY C

C_plus_comm_thm

$$\vdash \forall x y \bullet x + y = y + x$$

C_plus_assoc_thm

$$\vdash \forall x y z \bullet (x + y) + z = x + y + z$$

C_plus_assoc_thm1

$$\vdash \forall x y z \bullet x + y + z = (x + y) + z$$

C_plus_0_thm $\vdash \forall x \bullet x + \text{NC } 0 = x \wedge \text{NC } 0 + x = x$

C_plus_minus_thm

$$\vdash \forall x \bullet x + \sim x = \text{NC } 0 \wedge \sim x + x = \text{NC } 0$$

C_times_conj_thm

$$\vdash \forall x \bullet x * x^- = \text{RC } (\text{Re } x^{\wedge 2} + \text{Im } x^{\wedge 2})$$

C_times_comm_thm

$$\vdash \forall x y \bullet x * y = y * x$$

C_times_assoc_thm

$$\vdash \forall x y z \bullet (x * y) * z = x * y * z$$

C_times_assoc_thm1

$$\vdash \forall x y z \bullet x * y * z = (x * y) * z$$

C_times_1_thm

$$\vdash \forall x \bullet x * \text{NC } 1 = x \wedge \text{NC } 1 * x = x$$

C_times_recip_thm

$$\vdash \forall x \bullet \neg x = \text{NC } 0 \Rightarrow x * x^{-1} = \text{NC } 1$$

C_times_recip_thm1

$$\vdash \forall x \bullet \neg x = \text{NC } 0 \Rightarrow x^{-1} * x = \text{NC } 1$$

C_times_eq_1_thm

$$\vdash \forall z w \bullet \neg z = \text{NC } 0 \wedge z * w = \text{NC } 1 \Rightarrow w = z^{-1}$$

C_times_plus_distrib_thm

$$\begin{aligned} \vdash \forall x y z \\ \bullet x * (y + z) = x * y + x * z \\ \wedge (x + y) * z = x * z + y * z \end{aligned}$$

C_conj_plus_homomorphism_thm

$$\vdash \forall x y \bullet (x + y)^- = x^- + y^-$$

C_conj_times_homomorphism_thm

$$\vdash \forall x y \bullet (x * y)^- = x^- * y^-$$

RC_plus_homomorphism_thm

$$\vdash \forall x y \bullet \text{RC } (x + y) = \text{RC } x + \text{RC } y$$

RC_times_homomorphism_thm

$$\vdash \forall x y \bullet \text{RC } (x * y) = \text{RC } x * \text{RC } y$$

NC_plus_homomorphism_thm

$$\vdash \forall x y \bullet \text{NC } (x + y) = \text{NC } x + \text{NC } y$$

NC_one_one_thm

$$\vdash \forall m n \bullet \text{NC } m = \text{NC } n \Leftrightarrow m = n$$

NC_times_homomorphism_thm

$$\vdash \forall x y \bullet \text{NC } (x * y) = \text{NC } x * \text{NC } y$$

C_plus_order_thm

$$\begin{aligned} \vdash \forall x y z \\ \bullet y + x = x + y \\ \wedge (x + y) + z = x + y + z \\ \wedge y + x + z = x + y + z \end{aligned}$$

R_eq_thm

$$\vdash \forall x y \bullet x = y \Leftrightarrow x + \sim y = \text{NC } 0$$

C_minus_clauses

- $$\vdash \forall x y$$
- $\sim(\sim x) = x$
 $\wedge x + \sim x = \text{NC } 0$
 $\wedge \sim x + x = \text{NC } 0$
 $\wedge \sim(x + y) = \sim x + \sim y$
 $\wedge \sim(\text{NC } 0) = \text{NC } 0$

C_plus_clauses

- $$\vdash \forall x y z$$
- $(x + z = y + z \Leftrightarrow x = y)$
 $\wedge (z + x = y + z \Leftrightarrow x = y)$
 $\wedge (x + z = z + y \Leftrightarrow x = y)$
 $\wedge (z + x = z + y \Leftrightarrow x = y)$
 $\wedge (x + z = z \Leftrightarrow x = \text{NC } 0)$
 $\wedge (z + x = z \Leftrightarrow x = \text{NC } 0)$
 $\wedge (z = z + y \Leftrightarrow y = \text{NC } 0)$
 $\wedge (z = y + z \Leftrightarrow y = \text{NC } 0)$
 $\wedge x + \text{NC } 0 = x$
 $\wedge \text{NC } 0 + x = x$
 $\wedge \neg \text{NC } 1 = \text{NC } 0$
 $\wedge \neg \text{NC } 0 = \text{NC } 1$

C_times_order_thm

- $$\vdash \forall x y z$$
- $y * x = x * y$
 $\wedge (x * y) * z = x * y * z$
 $\wedge y * x * z = x * y * z$

C_times_0_thm

- $$\vdash \forall x \bullet x * \text{NC } 0 = \text{NC } 0 \wedge \text{NC } 0 * x = \text{NC } 0$$

C_times_eq_0_thm

- $$\vdash \forall x y \bullet x * y = \text{NC } 0 \Rightarrow x = \text{NC } 0 \vee y = \text{NC } 0$$

C_times_clauses

- $$\vdash \forall x$$
- $\text{NC } 0 * x = \text{NC } 0$
 $\wedge x * \text{NC } 0 = \text{NC } 0$
 $\wedge x * \text{NC } 1 = x$
 $\wedge \text{NC } 1 * x = x$

C_times_minus_thm

- $$\vdash \forall x y$$
- $\sim x * y = \sim(x * y)$
 $\wedge x * \sim y = \sim(x * y)$
 $\wedge \sim x * \sim y = x * y$

C_N_exp_1_thm

- $$\vdash \forall n \bullet \text{NC } 1 \wedge^n = \text{NC } 1$$

C_N_exp_rw_thm

- $$\vdash \forall z n$$
- $z \wedge^n = (\text{if } n = 0 \text{ then } \text{NC } 1 \text{ else } z * z \wedge^{(n-1)})$

C_N_exp_plus_thm

- $$\vdash \forall z m n \bullet z \wedge^{(m+n)} = z \wedge^m * z \wedge^n$$

C_N_exp_clauses

- $$\vdash \forall z n \bullet z \wedge^0 = \text{NC } 1 \wedge z \wedge^1 = z \wedge \text{NC } 1 \wedge^n = \text{NC } 1$$

de_moiivre_thm

$\vdash \forall x m$
 $\bullet (Cos\ x, Sin\ x) \wedge m$
 $= (Cos\ (\mathbb{NR}\ m * x), Sin\ (\mathbb{NR}\ m * x))$

C_series_rw_thm

$\vdash \forall s n$
 $\bullet Series_C\ s\ n$
 $= (if\ n = 0$
 $\quad then\ \mathbb{NC}\ 0$
 $\quad else\ s\ (n - 1) + Series_C\ s\ (n - 1))$

C_poly_eval_rec_thm

$\vdash (\forall s z \bullet PolyEval_C\ (s, 0)\ z = s\ 0)$
 $\wedge (\forall s n z$
 $\bullet PolyEval_C\ (s, n + 1)\ z$
 $= PolyEval_C\ (s, n)\ z$
 $+ s\ (n + 1) * z \wedge (n + 1))$

C_poly_eval_eq_thm1

$\vdash \forall s t m z$
 $\bullet (\forall i \bullet i \leq m \Rightarrow s\ i = t\ i)$
 $\Rightarrow PolyEval_C\ (s, m)\ z = PolyEval_C\ (t, m)\ z$

C_poly_eval_eq_thm2

$\vdash \forall s m n z$
 $\bullet (\forall i \bullet m < i \Rightarrow s\ i = \mathbb{NC}\ 0) \wedge m \leq n$
 $\Rightarrow PolyEval_C\ (s, m)\ z = PolyEval_C\ (s, n)\ z$

C_poly_eval_eq_thm

$\vdash \forall s t m n z$
 $\bullet (\forall i \bullet i \leq m \Rightarrow s\ i = t\ i)$
 $\wedge (\forall i \bullet m < i \Rightarrow t\ i = \mathbb{NC}\ 0)$
 $\wedge m \leq n$
 $\Rightarrow PolyEval_C\ (s, m)\ z = PolyEval_C\ (t, n)\ z$

C_poly_eval_0_thm

$\vdash \forall n z \bullet PolyEval_C\ ((\lambda i \bullet \mathbb{NC}\ 0), n)\ z = \mathbb{NC}\ 0$

C_poly_eval_minus_thm

$\vdash \forall s n z$
 $\bullet PolyEval_C\ ((\lambda i \bullet \sim (s\ i)), n)\ z$
 $= \sim (PolyEval_C\ (s, n)\ z)$

C_poly_eval_plus_thm

$\vdash \forall s t n z$
 $\bullet PolyEval_C\ ((\lambda i \bullet s\ i + t\ i), n)\ z$
 $= PolyEval_C\ (s, n)\ z + PolyEval_C\ (t, n)\ z$

C_const_eval_thm

$\vdash \forall c \bullet (\lambda x \bullet c) = PolyEval_C\ ((\lambda i \bullet c), 0)$

C_id_eval_thm

$\vdash (\lambda x \bullet x)$
 $= PolyEval_C$
 $((\lambda i \bullet if\ i = 1\ then\ \mathbb{NC}\ 1\ else\ \mathbb{NC}\ 0), 1)$

C_plus_eval_thm

$\vdash \forall s m t n$
 $\bullet (\lambda x \bullet PolyEval_C\ (s, m)\ x + PolyEval_C\ (t, n)\ x)$
 $= PolyEval_C\ (PlusCoeffs_C\ (s, m)\ (t, n))$

C_plus_eval_rw_thm

$\vdash \forall sm tn$

• $PolyEval_C (PlusCoeffs_C sm tn)$
 $= (\lambda x \bullet PolyEval_C sm x + PolyEval_C tn x)$

$\mathbb{C}_{const_times_eval_thm}$
 $\vdash \forall c s m$
 • $PolyEval_C ((\lambda i \bullet c * s i), m)$
 $= (\lambda x \bullet c * PolyEval_C (s, m) x)$

$\mathbb{C}_{times_eval_thm}$
 $\vdash \forall s m t n$
 • $(\lambda x \bullet PolyEval_C (s, m) x * PolyEval_C (t, n) x)$
 $= PolyEval_C (TimesCoeffs_C (s, m) (t, n))$

$\mathbb{C}_{times_eval_rw_thm}$
 $\vdash \forall sm tn$
 • $PolyEval_C (TimesCoeffs_C sm tn)$
 $= (\lambda x \bullet PolyEval_C sm x * PolyEval_C tn x)$

$\mathbb{C}_{poly_induction_thm}$
 $\vdash \forall p$
 • $(\forall c \bullet p (\lambda x \bullet c))$
 $\wedge p (\lambda x \bullet x)$
 $\wedge (\forall f g \bullet p f \wedge p g \Rightarrow p (\lambda x \bullet f x + g x))$
 $\wedge (\forall f g \bullet p f \wedge p g \Rightarrow p (\lambda x \bullet f x * g x))$
 $\Rightarrow (\forall h \bullet h \in PolyFunc_C \Rightarrow p h)$

$\mathbb{C}_{poly_func_eq_poly_eval_thm}$
 $\vdash PolyFunc_C = \{f | \exists s n \bullet f = PolyEval_C (s, n)\}$

$open_C_topology_thm$
 $\vdash O_C \in Topology$

$open_C_hausdorff_thm$
 $\vdash O_C \in Hausdorff$

$universe_open_C_thm$
 $\vdash Universe \in O_C$

$space_t_open_C_thm$
 $\vdash Space_T O_C = Universe$

$space_t_subspace_open_C_thm$
 $\vdash \forall A \bullet Space_T (A \triangleleft_T O_C) = A$

$\mathbb{C}_{minus_continuous_thm}$
 $\vdash \sim \in (O_C, O_C) Continuous$

$\mathbb{C}_{conj_continuous_thm}$
 $\vdash \$- \in (O_C, O_C) Continuous$

$\mathbb{C}_{plus_continuous_thm}$
 $\vdash Uncurry \$+ \in (O_C \times_T O_C, O_C) Continuous$

$\mathbb{C}_{times_continuous_thm}$
 $\vdash Uncurry \$* \in (O_C \times_T O_C, O_C) Continuous$

$\mathbb{C}_N_exp_continuous_thm$
 $\vdash \forall n \bullet (\lambda x \bullet x \wedge n) \in (O_C, O_C) Continuous$

$\mathbb{C}_{poly_continuous_thm}$
 $\vdash \forall f \bullet f \in PolyFunc_C \Rightarrow f \in (O_C, O_C) Continuous$

$\mathbb{C}_{series_continuous_thm}$
 $\vdash \forall \sigma s n$
 • $\sigma \in Topology \wedge (\forall i \bullet s i \in (\sigma, O_C) Continuous)$
 $\Rightarrow (\lambda x \bullet Series_C (\lambda i \bullet s i x) n)$
 $\in (\sigma, O_C) Continuous$

$exp_s1_continuous_thm$

$\vdash \text{ExpS1} \in (O_R, O_C) \text{ Continuous}$
C_abs_squared_lemma
 $\vdash \forall x y \bullet \text{Sqrt} (x^2 + y^2) = 1. \Leftrightarrow x^2 + y^2 = 1.$
∈_s1_lemma
 $\vdash \forall z \bullet z \in S1 \Leftrightarrow \text{Re } z^2 + \text{Im } z^2 = 1.$
exp_exp_s1_thm
 $\vdash \text{ExpS1} = \text{Exp} \circ \mathbb{R}i$
exp_s1_homomorphism_thm
 $\vdash \forall x y \bullet \text{ExpS1} (x + y) = \text{ExpS1 } x * \text{ExpS1 } y$
exp_s1_homomorphism_thm1
 $\vdash \forall x y \bullet \text{ExpS1 } x * \text{ExpS1 } y = \text{ExpS1} (x + y)$
exp_s1_minus_thm
 $\vdash \forall x \bullet \text{ExpS1} (\sim x) = \text{ExpS1 } x^{-1}$
exp_s1_∈_s1_thm
 $\vdash \forall x \bullet \text{ExpS1 } x \in S1$
exp_s1_period_thm
 $\vdash \forall x y$

- $\text{ExpS1 } x = \text{ExpS1 } y$
- $\Leftrightarrow (\exists m$
 - $y = x + \text{NR} (2 * m) * \pi$
 - $\vee x = y + \text{NR} (2 * m) * \pi)$

exp_s1_onto_thm
 $\vdash \forall z$

- $z \in S1 \Rightarrow (\exists_1 x \bullet 0. \leq x \wedge x < 2. * \pi \wedge z = \text{ExpS1 } x)$

s1_times_thm
 $\vdash \forall z w \bullet z \in S1 \wedge w \in S1 \Rightarrow z * w \in S1$
exp_s1_onto_thm1
 $\vdash \forall c z$

- $z \in S1$
- $\Rightarrow (\exists_1 x \bullet c - \pi \leq x \wedge x < c + \pi \wedge z = \text{ExpS1 } x)$

exp_s1_onto_thm2
 $\vdash \forall c z$

- $z \in S1 \wedge \neg z = \text{ExpS1} (c + \pi)$
- $\Rightarrow (\exists_1 x$
 - $x \in \text{OpenInterval} (c - \pi) (c + \pi) \wedge z = \text{ExpS1 } x)$

exp_s1_covering_projection_lemma1
 $\vdash \forall c$

- $S1 \setminus \{\text{ExpS1} (c + \pi)\}$
- $= \{z$
 - $| \exists x$
 - $x \in \text{OpenInterval} (c - \pi) (c + \pi)$
 - $\wedge z = \text{ExpS1 } x\}$

exp_s1_covering_projection_lemma2
 $\vdash \forall x$

- ExpS1
- $\in (\text{OpenInterval} (x - \pi) (x + \pi) \triangleleft_T O_R,$
- $(S1 \setminus \{\text{ExpS1} (x + \pi)\}) \triangleleft_T O_C) \text{ Homeomorphism}$

exp_s1_covering_projection_lemma3
 $\vdash \text{ExpS1} \in (O_R, S1 \triangleleft_T O_C) \text{ Continuous}$
exp_s1_covering_projection_lemma4
 $\vdash \forall x \bullet \text{ExpS1 } x \in S1 \setminus \{\text{ExpS1} (x + \pi)\}$
exp_s1_covering_projection_lemma5
 $\vdash \forall x y$

$$\begin{aligned}
& \bullet \text{ExpS1 } x = \text{ExpS1 } y \\
& \quad \wedge \neg \text{OpenInterval } (x + \sim \pi) (x + \pi) \\
& \quad \quad \cap \text{OpenInterval } (y + \sim \pi) (y + \pi) \\
& \quad = \{\} \\
& \Rightarrow x = y
\end{aligned}$$

C_punctured_set_thm

$$\vdash \forall X x \bullet X \setminus \{x\} \in X \triangleleft_T O_C$$

exp_s1_covering_projection_thm

$$\vdash \text{ExpS1} \in (O_R, S1 \triangleleft_T O_C) \text{CoveringProjection}$$

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