Mathematical Case Studies:
—
Some Topology*

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Abstract

This ProofPower-HOL document contains definitions and proofs concerning some basics of abstract topology, metric space theory and algebraic topology (more precisely, elementary homotopy theory). It presents the material using the approach taken in [1]: the main body of the document contains the definitions together with a narrative commentary including a discussion of the theorems that have been proved. This is followed by a listing of the theory and an index to the theorems and definitions. The source text of this document also contains the proof scripts, but these are suppressed from the printed form by default.

The coverage of abstract topology includes the definitions of the following: topologies; construction of new topologies from old as (binary) product spaces or subspaces; continuity, Hausdorff spaces; connectedness; compactness, the interior, boundary and closure operators; a notion of *protocomplex* that we later use to define CW complexes. A range of basic theorems are proved, including: continuity of functional composition and of the structural maps for products; preservation of compactness and connectedness under continuous maps; connectedness resp. compactness of products of connected resp. compact spaces.

The coverage of metric spaces is very minimal. The standard arguments in the algebraic topology we are interested in can be done with almost no metric space ideas. The main idea that is needed is the notion of the Lebesgue number of a covering (which is needed to show that if you cover an interval or a square with open sets, then on some suitably fine subdivision of the interval or square, each subinterval or grid cell is contained in one of the covering sets). With these applications in view, the metrics for the real line and the plane are defined. We also define euclidean n-space in general using lists of reals for the representation and use these to define cubes, spheres and CW complexes. (Technical note: we actually use the L_1 (Manhattan taxi-cab) metric on product spaces, not the more usual L_2 (Euclidean) metric. The L_1 metric gives the same topology and makes the arithmetic easier in most cases.)

Finally, we deal with some basics of homotopy theory. This material is very far from complete. Currently we have: the definition of path connectedness and the proof that path connected spaces are connected; definitions of the notions of homotopy and of homotopy classes with the proofs that the homotopy relation is an equivalence relation; definitions of the path space (qua set, not qua space, in fact) together with the of the operations that induce a groupoid structure on the homotopy classes in the path space together with the proofs that these operations do indeed give a groupoid modulo homotopy equivalence; definition of a covering projection and a proof that covering projections enjoy the unique lifting property and the homotopy lifting property.

Copyright © : Lemma 1 Ltd 2004–2016
Reference: LEMMA1/HOL/WRK067; Current git revision: b0522be

*First posted 11 April 2004; for full changes history see: `https://github.com/RobArthan/pp-contrib`.

Contents

To Do

This document is a somewhat inchoate state pending some more work on algebra to inform the work on elementary homotopy theory.

- Add some more references.
- Some of the results in the theory of the real line and the plane belong logically in the theory of metric spaces. Move them into the right place.
- Make the treatment of homotopy classes use the treatment of equivalence relations in [2].
- Complete the work on homotopy theory at least to the point where the fundamental group can be defined and said to be a group in the sense of [2] (and calculate some non-trivial fundamental groups, e.g., that of the circle).

Acknowledgments

Thanks to Bill Richter for pointing out that one can define the notion of homotopy lifting property with respect to a given space without falling foul of the restriction on type variables in constant specifications.

References

[1] LEMMA1/HOL/WRK066. *Mathematical Case Studies: Basic Analysis*. R.D. Arthan, Lemma 1 Ltd., `rda@lemma-one.com`.

[2] LEMMA1/HOL/WRK068. *Mathematical Case Studies: Some Group Theory*. R.D. Arthan, Lemma 1 Ltd., `rda@lemma-one.com`.

1 ABSTRACT TOPOLOGY

1.1 Technical Prelude

The following ML commands set up a theory "topology" to hold the definitions and theorems and set up a convenient proof context. The parents of the theory are the theory "bin_rel" of binary relations and the theory "fincomb" of finite combinatorics.

SML

```
force_delete_theory"topology" handle Fail _ => ();
open_theory"bin_rel";
set_merge_pcs["basic_hol1", "'sets_alg"];
new_theory"topology";
new_parent"fincomb";
```

1.2 Topologies

We begin with the definition of a topology. We follow the most common tradition of defining a topology by specifying its set of open sets. The polymorphic set *Topology* is the set of all sets of sets that we will consider to be topologies. We do not require a topology to form a topology on the universe of the type of its points. For example, we wish to consider sets such as the unit interval in the real line to be topological spaces in their own right. This actually simplifies the definition: we just require a topology to be a set of sets that is closed under arbitrary unions and binary intersections. We do not require the carrier set of the topology to be a non-empty set (as some elementary text books do, unnecessarily). Nor do we need to make a special case of the empty set — it will be shown to be an open set in any topology (as the union of an empty set of open sets).

SML

HOL Constant

Topology : 'a SET SET SET

$$Topology = \{\tau \mid (\forall V\bullet\ V \subseteq \tau \Rightarrow \bigcup V \in \tau) \land (\forall A\ B\bullet A \in \tau \land B \in \tau \Rightarrow A \cap B \in \tau)\}$$

We can recover the carrier set of a topology as the union of all its open sets. It reads quite nicely to call this the *space* of the topology.

SML

HOL Constant

$\boldsymbol{Space_T}$: 'a SET SET → 'a SET

$$\forall \tau\bullet\ Space_T\ \tau = \bigcup \tau$$

A set is *closed* with respect to a topology, τ, if it is the complement of an open set (i.e., a member of τ) relative to the space of τ. For this and several other concepts, we use postfix notation to suggest informal notations like "τ−closed".

SML

```
declare_postfix(400, "Closed");
```

HOL Constant

$\$\boldsymbol{Closed} : 'a\ SET\ SET \to 'a\ SET\ SET$

$\forall \tau \bullet\ \tau\ Closed = \{A \mid \exists B \bullet B \in \tau \land A = Space_T\ \tau \setminus B\}$

Note that in the above definition and in those that follow, we do no stipulate that τ actually be a topology. We will agree, however, in stating theorems, to make that an assumption whenever necessary (which is nearly always in theorems of any interest).

The theorems begin with some preliminary lemmas about enumerated sets, finite sets and maxima and minima that belong elsewhere eventually.

enum_set_⊆_thm
⋃_enum_set_clauses
⋂_enum_set_clauses
finite_image_thm
⊆_size_thm

Now comes a batch of useful basic facts about open and closed sets: the empty set and the space of a topology are both open and both closed; a set is open iff it contains an open neighbourhood of each of its points; a set is closed iff its complement contains an open neighbourhood of each of its points; any member of an open set is a member of the space (a technical convenience in later proofs); binary unions and, more generally, arbitrary unions of open sets are open; binary intersections and, more generally, finite intersections of open sets are open.

empty_open_thm
space_t_open_thm
empty_closed_thm
space_t_closed_thm
open_open_neighbourhood_thm
closed_open_neighbourhood_thm
∈_space_t_thm
∪_open_thm
⋃_open_thm
∩_open_thm
finite_⋂_open_thm

1.3 New Topologies from Old: Subspace and Product Topologies

We borrow the Z symbol for range restriction (decorated with a subscript to avoid overloading) for the operator that forms the subspace of a topological space defined by some subset of the universe of its points. If that subset contains points outside the carrier set of the topological space they are ignored. A set is open with respect to the subspace topology defined by a subset X of the space of the topology iff it is the intersection of an open set with X.

SML

```
declare_infix(280, "◁T");
```

HOL Constant

$$\mathbf{\$\lhd_T} : 'a\ SET \to 'a\ SET\ SET \to 'a\ SET\ SET$$

$$\forall X\ \tau\bullet \quad (X \lhd_T \tau) = \{A \mid \exists B\bullet\ B \in \tau \land A = B \cap X\}$$

We now give basic facts about the subspace topology induced by a subset of the space of a topology: it is a topology; its space is the subset; if the subset is the space of the topology, the subspace topology and the original are the same.

subspace_topology_thm *subspace_topology_space_t_thm* *trivial_subspace_topology_thm*

The definition of the (binary) product topology is the usual one (which amounts to saying that the sets of the form $A \times B$ where A and B are open in the factors of the product provide a basis for the product topology).

SML

```
declare_infix(290, "×T");
```

HOL Constant

$$\mathbf{\$\times_T} : 'a\ SET\ SET \to 'b\ SET\ SET \to ('a \times 'b)\ SET\ SET$$

$$\forall \sigma\ \tau\bullet\ (\sigma \times_T \tau) = \{C \mid \forall\ x\ y\bullet\ (x,\ y) \in C \Rightarrow \exists A\ B\bullet\ A \in \sigma \land B \in \tau \land x \in A \land y \in B \land (A \times B) \subseteq C\}$$

The product topology is indeed a product topology and the space of the product topology is the product of the spaces of the factors:

product_topology_thm *product_topology_space_t_thm*

The trivial topology on a 1-point type is useful.

HOL Constant

$$\mathbf{\$1_T} : ONE\ SET\ SET$$

$$1_T = \{\{\};\ \{One\}\}$$

unit_topology_thm *unit_topology_space_t_thm*

Now we define the n-th power topology for finite n: if the space of τ is X $\Pi_T\ n\ \tau$ is the usual topology on X^n.

HOL Constant

$\mathbf{\$\Pi_T} : \mathbb{N} \to {}'a\ SET\ SET \to {}'a\ LIST\ SET\ SET$

$$\forall \tau\ n \bullet \quad \Pi_T\ 0\ \tau = \{\ \{\}; \{[]\}\ \}$$
$$\wedge \quad (\Pi_T\ (n{+}1)\ \tau) = \{C \mid \neg[] \in C \wedge \forall\ x\ v \bullet\ Cons\ x\ v \in C \Rightarrow$$
$$\exists A\ B \bullet\ A \in \tau \wedge B \in \Pi_T\ n\ \tau \wedge x \in A \wedge v \in B \wedge$$
$$\forall y\ w \bullet y \in A \wedge w \in B \Rightarrow Cons\ y\ w \in C\}$$

Apart from the easy lemma which says that the lists in $\Pi_T\ n\ \tau$ are all of length n and the fact that the power topology is a topology, we defer proofs about the power topology until we have defined homeomorphisms.

power_topology_length_thm *power_topology_thm*

1.4 Continuity

There are some issues about the precise formalisation of continuity. The interesting part is completely standard: a function is continuous iff the inverse images of open sets are open sets. Clearly, there are two topologies here: one for the domain of the function and one for its range. It is technically convenient to work with functions that are total on the universe of the type of the domain. In any case, we want to support something like the usual way of thinking in the calculus where one doesn't carefully restrict every function to the domain of interest. E.g., one says things like "1/sin x is continuous from $(0, \pi/2)$ to the positive real numbers.".

The upshot is the following definition of a continuous function from the topological space σ to the topological space τ. The function is required to map the carrier set of σ to that of τ. It may well also map things outside the carrier set of σ into that of τ, and these need to be filtered out when we are testing whether the inverse image of an open set is open.

SML

```
declare_postfix(400, "Continuous");
```

HOL Constant

$\mathbf{\$Continuous} : ({}'a\ SET\ SET \times {}'b\ SET\ SET) \to ({}'a \to {}'b)\ SET$

$$\forall \sigma\ \tau \bullet\ (\sigma, \tau)\ Continuous =$$
$$\{f$$
$$\mid \quad (\forall x \bullet\ x \in Space_T\ \sigma \Rightarrow f\ x \in Space_T\ \tau)$$
$$\wedge \quad (\forall A \bullet\ A \in \tau \Rightarrow \{x \mid x \in Space_T\ \sigma \wedge f\ x \in A\} \in \sigma)\}$$

We now give some principles for recognising continuous functions. First of all a function is continuous iff the inverse image of each closed set is closed. The restriction of a continuous function to a subspace is continuous. The following are all continuous: constant functions, identity functions, compositions of continuous functions, the projections of a product onto its factors, the pointwise product of two continuous functions with common domain, the natural injections of a factor of a product into the product, the inclusion of the diagonal into the product of a topological space with itself, a function whose domain or range is the unit topological space, and, finally, a function defined by cases under suitable hypotheses. The last-mentioned principle says that, given two continuous functions, f and g, on the same topological space and a subset, X, of their domain, the function that agrees with f on X and with g elsewhere is continuous provided f and g agree on each point which lies both in the closure of X and in the closure of its complement.

continuous_closed_thm
subspace_continuous_thm
const_continuous_thm
id_continuous_thm
comp_continuous_thm
left_proj_continuous_thm
right_proj_continuous_thm
product_continuous_thm
product_continuous_⇔_thm
left_product_inj_continuous_thm
right_product_inj_continuous_thm
domain_unit_topology_continuous_thm
range_unit_topology_continuous_thm
diag_inj_continuous_thm
cond_continuous_thm

1.5 Hausforff Separation Condition

Now we define the Hausdorff separation condition. A topology is Hausdorff iff any two distinct elements possess disjoint open neighbourhoods.

SML

HOL Constant

$\boldsymbol{Hausdorff} : {'a}\ SET\ SET\ SET$

$Hausdorff =$
$\{\tau \mid \forall x\ y \bullet\ x \in Space_T\ \tau \land y \in Space_T\ \tau \land \neg x = y$
$\Rightarrow \quad \exists A\ B \bullet A \in \tau \land B \in \tau \land x \in A \land y \in B \land A \cap B = \{\}\}$

A subspace of a Hausdorff space is Hausdorff as is the product of two Hausdorff spaces:

subspace_topology_hausdorff_thm
product_topology_hausdorff_thm

1.6 Compactness

The definition of compactness is the standard one (a topology is compact iff every open covering has a finite subcovering), together with the explicit requirement that the compact set be a subset of the space of the topology in question.

SML

```
declare_postfix(400, "Compact");
```

HOL Constant

$$\$\mathbf{Compact} : 'a\ SET\ SET \to 'a\ SET\ SET$$

$$\forall \tau \bullet\ \tau\ Compact = \{A \mid A \subseteq Space_T\ \tau \land \forall V \bullet\ V \subseteq \tau \land A \subseteq \bigcup V \Rightarrow \exists W \bullet\ W \subseteq V \land W \in Finite \land A \subseteq \bigcup W\}$$

Compactness is a topological property, i.e., compactness of a set depends only on the topology induced on the set and not on how the set is embedded in the containing topological space; continuous functions map compact sets to compact sets; the union of two compact sets is again compact; a compact subset of a Hausdorff space is closed. The final result is preceded by a simple lemma about separating a point from the union of a finite set of sets.

compact_topological_thm
image_compact_thm
∪_compact_thm
compact_closed_lemma
compact_closed_thm

Now we show that the product of two compact sets is compact. This is the finite case of Tychonov's theorem. The proof in the finite case is much simpler than the general case. Moreover the general case is probably best stated in terms of a topology on a function space and we do not wish to consider such topologies yet. We sneak up on the proof in three steps: the first two are of general use: *compact_basis_thm* says that given a basis for a topology, to check compactness of a set one only needs to consider coverings by basic open sets and *compact_basis_product_topology_thm* is the special case of this where the topology is the product topology and the basis is the basis that defines the product topology. *compact_product_lemma* is a somewhat ad hoc lemma that is needed in the proof of the main theorem and might be of use elsewhere.

compact_basis_thm
compact_basis_product_topology_thm
compact_product_lemma
product_compact_thm

Finally, for use in producing Lebesgue numbers of coverings of compact subsets of metric spaces, we prove that compact sets are sequentially compact (every countable subset has a limit point). We precede the proof by a lemma saying that if a (countably infinite) sequence ranges over the union of a finite family of sets, then some member of the family is visited infinitely often.

compact_sequentially_compact_lemma
compact_sequentially_compact_thm

1.7 Connectedness

Similarly, the definition of connectedness is the standard one (a topology is connected if its space cannot be written as the union of two disjoint open sets), again together with the explicit requirement that the connected set be a subset of the carrier set of the topology in question.

SML

```
declare_postfix(400, "Connected");
```

HOL Constant

$$\mathbf{\$Connected} : 'a\ SET\ SET \to 'a\ SET\ SET$$

$$\forall \tau \bullet\ \tau\ Connected = \{A \mid A \subseteq Space_T\ \tau \land \forall B\ C \bullet\ B \in \tau \land C \in \tau \land A \subseteq B \cup C \land A \cap B \cap C = \{\} \Rightarrow (A \subseteq B \lor A \subseteq C)\}$$

Connectedness is a topological property[1]. a set is connected iff it cannot be separated by two closed sets; a set is connected iff any two of its points are contained in a connected subset of the set (which doesn't sound very useful, but is, so much so that we present it both as a conditional rewrite rule and in a form suitable for back-chaining);

connected_topological_thm	*connected_pointwise_thm*
connected_closed_thm	*connected_pointwise_bc_thm*

The empty set is connected as is any singleton set; continuous functions map connected sets to connected sets; the union of two non-disjoint connected sets is connected as is the product of any two connected sets. If the union of two non-empty open (or closed) sets is connected the two sets cannot be disjoint.

empty_connected_thm	∪_*connected_thm*	∪_*open_connected_thm*
singleton_connected_thm	*product_connected_thm*	∪_*closed_connected_thm*
image_connected_thm		

Results of the following sort capture common ways of thinking about spaces such as geometric simplicial complexes or CW complexes constructed by gluing together connected pieces:

- the union of three connected sets is connected if they can be listed, so that each member meets the next member in the list;
- if a connected set is covered by a set of connected sets, then the union of the covering sets is itself connected;
- if the union of two connected sets is not connected, then the two sets can be separated (by two open sets, which may not be disjoint in general, but are each disjoint from the union);
- if a connected set can be separated from each of a finite family of connected sets, then it can be separated from the union of the family;
- given a finite family of non-empty connected sets U and a connected set B such that B is connected as is the union of B and the sets in U, if B does not contain every set in U, then there is some set A in U such that the union of A and B is connected;
- given a finite family of non-empty connected sets U and a member A of U, one can begin with A and deal out sets from U such that at each stage the union of the sets that have been dealt

[1] The use of $A \cap B \cap C = \{\}$ rather than $B \cap C = \{\}$ in the definition is perhaps surprising, but connectedness would not be a topological property with the latter formulation. To see this, consider a space X with three points x, y and z, topologised so that O is open iff $O = \{\}$ or $z \in O$. Then x and y cannot be separated by disjoint open sets in X, but $\{x, y\}$ is not connected under the subspace topology.

is connected, such that each set dealt adds to this union whenever that is possible, and such that eventually the union of the sets that have been dealt is equal to the union of all the sets in U;

- given a finite family of non-empty connected sets U and a member A of U, either A contains the union of all the sets in U, or there is a B in U not equal to A and such that the union of the sets in U other than B is connected and does not contain B.

∪_∪_connected_thm
cover_connected_thm
separation_thm
finite_separation_thm
connected_extension_thm
connected_chain_thm
connected_step_thm

1.8 Homeomorphisms

A homeomorphism is a continuous mapping with a continuous two-sided inverse:

SML

```
declare_postfix(400, "Homeomorphism");
```

HOL Constant

$Homeomorphism : ('a SET SET × 'b SET SET) → ('a → 'b) SET

```
∀σ τ• (σ, τ) Homeomorphism =
    {f
    |       f ∈ (σ, τ) Continuous
    ∧       ∃g•     g ∈ (τ, σ) Continuous
            ∧       (∀x•x ∈ Space_T σ ⇒ g(f x) = x)
            ∧       (∀y•y ∈ Space_T τ ⇒ f(g y) = y)}
```

The identity function is a homeomorphism as is the composite of two homeomorphisms, the product of a pair of homeomorphisms, the natural mapping from a space and its product with a one-point space and the function on product that interchanges the factors; a homeomorphism is an open mapping (i.e., it sends open sets to open sets) and is also one-to-one; a function is a homeomorphism iff it is a one-to-one, onto, continuous open mapping. Finally, a useful principle for recognising homeomorphisms obtained by restricting continuous functions defined on compact Hausdorff spaces.

id_homeomorphism_thm
comp_homeomorphism_thm
product_homeomorphism_thm
product_unit_homeomorphism_thm
swap_homeomorphism_thm
homeomorphism_open_mapping_thm
homeomorphism_one_one_thm
homeomorphism_onto_thm
homeomorphism_one_one_open_mapping_thm
⊆_compact_homeomorphism_thm

The useful principle is this: *Let C and X be Hausdorff spaces with C compact and let f be a continuous function from C to X. If $B \subseteq C$ is such that for every $y \in f(B)$ there is a unique x in C such that $f(x) = y$, then f restricts to a homeomorphism between B and $f(B)$.* To see this, note that it is enough to prove that the restriction of f to B is a closed mapping, since evidently f is one-one and continuous on B. Given a closed subset A of B, we have $A = B \cap D$ where D is some closed and hence compact subset of C. By assumption $f(D \backslash B)$ is disjoint from $f(B)$, which implies that $f(D \cap B) = f(D) \cap f(B)$. Since D is compact, so also is $f(D)$, whence $f(D)$ is closed. Thus $f(A) = f(D) \cap f(B)$ is a closed subset of $f(B)$.

1.9 Interior, Boundary and Closure Operators

Our definitions of the interior, boundary and Closure operators are standard, but as we have to be explicit about the ambient topology, we take them to be infix operators, reflecting usages like "the τ-interior of A" that one might use when working with several different topologies on the same set.

SML

```
declare_infix(400, "Interior");
declare_infix(400, "Boundary");
declare_infix(400, "Closure");
```

The τ-interior of A comprises the points that lie in τ-open subsets of A; the τ-boundary comprises the points (of the space) all of whose open neighbourhoods meet both A and its complement; the τ-closure of A is the smallest τ-closed set containing (the points of the space belonging to) A.

HOL Constant

```
$Interior $Boundary $Closure: 'a SET SET → 'a SET → 'a SET
──────────────────────────────
∀τ A•
        τ Interior A = {x | ∃B• B ∈ τ ∧ x ∈ B ∧ B ⊆ A}
∧       τ Boundary A =
        {x | x ∈ Space_T τ ∧ ∀B• B ∈ τ ∧ x ∈ B ⇒ ¬B ∩ A = {} ∧ ¬B \ A = {}}
∧       τ Closure A = ⋂{B | B ∈ τ Closed ∧ A ∩ Space_T τ ⊆ B}
```

The interior and boundary of a set are subsets of the ambient space and the interior is a subset of the set; the boundary of a set is the complement of the union of its interior and the complement of its interior; the interior of the product of two sets is the product of their interiors; a set is open iff it is disjoint from its boundary and closed iff it contains its boundary; the interior of a set is the union of its open subsets; the closure of a set is the complement of the interior of its complement.

interior_boundary_⊆_space_t_thm	*open_⇔_disjoint_boundary_thm*
interior_⊆_thm	*closed_⇔_boundary_⊆_thm*
boundary_interior_thm	*interior_⋃_thm*
interior_×_thm	*closure_interior_complement_thm*

1.10 Covering Projections

Our definition of covering projection is completely standard: a continuous function is a covering projection if every point in its range has a neighbourhood C whose inverse image is a disjoint union of open sets each of which is mapped homeomorphically onto C.

SML

```
declare_postfix(400, "CoveringProjection");
```

HOL Constant

$$\textbf{\$CoveringProjection} : ('a\ SET\ SET \times 'b\ SET\ SET) \to ('a \to 'b)\ SET$$

$$
\begin{array}{l}
\forall \sigma\ \tau \bullet\ (\sigma,\ \tau)\ CoveringProjection\ = \\
\quad \{p \\
\quad |\quad p \in (\sigma,\ \tau)\ Continuous \\
\quad \land\quad \forall y \bullet\quad y \in Space_T\ \tau \\
\qquad\quad \Rightarrow\quad \exists C \bullet\quad y \in C \land C \in \tau \land \\
\qquad\qquad \exists U \bullet\quad U \subseteq \sigma \\
\qquad\qquad \land\quad (\forall x \bullet\ x \in Space_T\ \sigma \land p\ x \in C \\
\qquad\qquad\qquad \Rightarrow \exists A \bullet\ x \in A \land A \in U) \\
\qquad\qquad \land\quad (\forall A\ B \bullet\ A \in U \land B \in U \land \neg A \cap B = \{\} \Rightarrow A = B) \\
\qquad\qquad \land\quad (\forall A \bullet\ A \in U \Rightarrow p \in (A \lhd_T \sigma,\ C \lhd_T \tau)\ Homeomorphism)\}
\end{array}
$$

We prove two lemmas that fit together to give the unique lifting property for continuous functions from a connected space into the base space of a covering projection.

unique_lifting_lemma1
unique_lifting_lemma2
unique_lifting_thm

1.11 Protocomplexes

In a later version of this document we intend to define the notion of a CW complex. To support this, it is convenient to define some purely topological notions. A *protocomplex* will comprise a set of pairs representing a partial function from certain closed subsets of a topological space X to the natural numbers. The sets in the domain of this function will be referred to as *cells* and the natural number associated with a cell will be called its *dimension*. Informally, we call a cell of dimension m an m-cell. The union of all the cells is the *space* of the protocomplex:

HOL Constant

$$\boldsymbol{Space_K} : ('a\ SET \times \mathbb{N})\ SET \to 'a\ SET$$

$$\forall C \bullet\ Space_K\ C = \bigcup \{c \mid \exists m \bullet\ (c,\ m) \in C\}$$

(We distinguish the name with a subscript K as in the German *Komplex*, since we use C elsewhere for the complex numbers.)

We define the n-skeleton of C to be the union of all cells of dimension at most n.

SML

```
declare_infix(400, "Skeleton");
```

HOL Constant

$$\textbf{\$Skeleton} : \mathbb{N} \to ('a\ SET \times \mathbb{N})\ SET \to 'a\ SET$$

$$\forall n\ C\bullet\ n\ Skeleton\ C = \bigcup\{c \mid \exists m\bullet m \le n \land (c,\ m) \in C\}$$

Our requirements on a protocomplex are as follows: *(i)* each cell is a closed set, *(ii)* for every x in X there is a unique m-cell c such that x lies in the interior of c with respect to the relative topology on the m-skeleton of C, *(iii)* a subset A of X is closed if $A \cap c$ is closed for every cell c, and *(iv)* each m-cell meets only finitely many cells of lower dimension.

HOL Constant

$$\textbf{Protocomplex} : 'a\ SET\ SET \to ('a\ SET \times \mathbb{N})\ SET\ SET$$

$$\begin{array}{l}
\forall C\ \tau\bullet C \in Protocomplex\ \tau \Leftrightarrow \\
\quad (\forall c\ m\bullet\ (c,\ m) \in C \Rightarrow c \in \tau\ Closed) \\
\land \quad (\forall x\bullet\ x \in Space_K\ C \Rightarrow \\
\qquad \exists_1\ (c,\ m)\bullet\ (c,\ m) \in C \land x \in ((m\ Skeleton\ C) \lhd_T \tau)\ Interior\ c) \\
\land \quad (\forall A\bullet\ A \subseteq Space_K\ C \land (\forall c\ m\bullet\ (c,\ m) \in C \Rightarrow A \cap c \in \tau\ Closed) \Rightarrow A \in \tau\ Closed) \\
\land \quad (\forall c\ m\bullet\ (c,\ m) \in C \Rightarrow \{(d,\ n) \mid (d,\ n) \in C \land n < m \land \neg c \cap d = \{\}\} \in Finite)
\end{array}$$

2 METRIC SPACES — DEFINITIONS

In the following, we bring in the theory of analysis from [1], although we could make do just with the real numbers to start with.

SML

```
force_delete_theory"metric_spaces" handle Fail _ => ();
open_theory"topology";
new_theory"metric_spaces";
new_parent"analysis";
new_parent"trees";
set_merge_pcs["basic_hol1", "'sets_alg", "'ℤ", "'ℝ"];
```

Our treatment of metric spaces is very minimal. The main fact we are interested in will be that coverings of compact subsets of metric spaces have a Lebesgue number. The definitions involved are the concept of a metric:

HOL Constant

$$\textbf{Metric} : ('a \times 'a \to \mathbb{R})\ SET$$

$$\begin{array}{lll}
 & Metric = & \\
 & \{ & D \\
 & | & (\forall x\ y\bullet\ \mathbb{NR}\ 0 \le D(x,\ y)) \\
 & \land & (\forall x\ y\bullet\ D(x,\ y) = \mathbb{NR}\ 0 \Leftrightarrow x = y) \\
 & \land & (\forall x\ y\bullet\ D(x,\ y) = D\ (y,\ x)) \\
 & \land & (\forall x\ y\ z\bullet\ D(x,\ z) \le D\ (x,\ y) + D(y,\ z))\}
\end{array}$$

...and the concept of the metric topology, which we write as a postfix since otherwise the notation for concepts such as "compact with respect to the metric topology induced by D" look rather strange.

SML

```
declare_postfix(400, "MetricTopology");
```

HOL Constant

$$\mathbf{\$MetricTopology} : ('a \times 'a \to \mathbb{R}) \to 'a\ SET\ SET$$

$$\forall D \bullet\ D\ MetricTopology = \{A \mid \forall x \bullet x \in A \Rightarrow \exists e \bullet\ \mathbb{NR}\ 0 < e \land (\forall y \bullet D(x,\ y) < e \Rightarrow y \in A)\}$$

We prove some basic facts about the metric topology and about the sum metric on a product of metric spaces.

metric_topology_thm
space_t_metric_topology_thm
open_ball_open_thm
open_ball_neighbourhood_thm
metric_topology_hausdorff_thm
product_metric_thm
product_metric_topology_thm

We prove the existence of Lebesgue numbers and that if X is a compact subset of an open space A in a metric space, then for small $\epsilon > 0$, A contains the ball $B(x, \epsilon)$ for every $x \in X$.

lebesgue_number_thm
collar_thm

We also define an induced metric on the set of lists of elements of a metric space. We use this, for example, to define n-dimensional euclidean space. Getting a good definition is a little delicate: given a (non-empty) metric space A with metric d, fix an arbitrary element $a \in A$ and let A^* be the set of countably infinite sequences in A that take the constant value a for all but finitely many indices. A^* becomes a metric space under the metric d^* defined by $d^*(s,t) = \sum_i d(s_i, t_i)$. If we map lists to infinite sequences by padding with a, this induces a pseudo-metric on the space A^L of lists of elements of A. To get a metric, we take $d^L(v,w) = |\#v - \#w| + d^*(vaaa\ldots, waaa\ldots)$, where $\#v$ is the length of the list v.

HOL Constant

$$\mathbf{ListMetric} : ('a \times 'a \to \mathbb{R}) \to ('a\ LIST \times 'a\ LIST) \to \mathbb{R}$$

$$\begin{aligned} &\forall D\ x\ v\ y\ w \bullet \\ &\qquad ListMetric\ D\ ([],\ []) = 0. \\ \land &\qquad ListMetric\ D\ (Cons\ x\ v,\ []) = 1. + D(x,\ Arbitrary) + ListMetric\ D\ (v,\ []) \\ \land &\qquad ListMetric\ D\ ([],\ Cons\ y\ w) = 1. + D(Arbitrary,\ y) + ListMetric\ D\ ([],\ w) \\ \land &\qquad ListMetric\ D\ (Cons\ x\ v,\ Cons\ y\ w) = D(x,\ y) + ListMetric\ D\ (v,\ w) \end{aligned}$$

list_pseudo_metric_lemma1
list_pseudo_metric_lemma2
list_metric_nonneg_thm
list_metric_sym_thm
list_metric_metric_thm

3 THE REAL LINE AND THE PLANE — DEFINITIONS

SML

```
force_delete_theory"topology_ℝ" handle Fail _ => ();
open_theory"metric_spaces";
new_theory"topology_ℝ";
set_merge_pcs["basic_hol1", "'sets_alg", "'ℤ", "'ℝ"];
```

We will make much use of the standard topology on the real line and so we define a short alias for it:

SML

```
declare_alias("O_R", ⌜Open_R⌝);
```

We define the standard metric on the real line:

HOL Constant

$$\boldsymbol{D_R} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$\forall x\ y \bullet\ D_R(x,\ y) = Abs(y - x)$$

In the plane, as we are primarily interested in topological properties it is simple and convenient to use the L_1-norm.

HOL Constant

$$\boldsymbol{D_{R2}} : (\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}) \to \mathbb{R}$$

$$\forall x1\ y1\ x2\ y2 \bullet\ D_{R2}\ ((x1,\ y1),\ (x2,\ y2)) = Abs(x2 - x1) + Abs(y2 - y1)$$

We

d_ℝ_2_def1
open_ℝ_topology_thm
space_t_ℝ_thm
closed_closed_ℝ_thm
compact_compact_ℝ_thm
continuous_cts_at_ℝ_thm
universe_ℝ_connected_thm
closed_interval_connected_thm
connected_ℝ_thm
continuous_ℝ_×_ℝ_ℝ_thm
continuous_ℝ_×_ℝ_ℝ_thm1
continuous_ℝ_×_ℝ_ℝ_thm3
continuous_ℝ_×_ℝ_ℝ_thm4
plus_continuous_ℝ_×_ℝ_thm
times_continuous_ℝ_×_ℝ_thm
cond_continuous_ℝ_thm
d_ℝ_metric_thm
d_ℝ_open_ℝ_thm
d_ℝ_2_metric_thm
d_ℝ_2_open_ℝ_×_open_ℝ_thm
open_ℝ_hausdorff_thm
open_ℝ_×_open_ℝ_hausdorff_thm
ℝ_lebesgue_number_thm
closed_interval_lebesgue_number_thm
product_interval_cover_thm
dissect_unit_interval_thm
product_interval_cover_thm

honour euclidean n-space with the name *Space* with no further decoration. For us, this is a family of topologies indexed by the natural numbers. The underlying spaces of the topologies comprise lists of real numbers.

SML

```
declare_postfix(400, "Space");
```

HOL Constant

$\boldsymbol{\$Space} : \mathbb{N} \to \mathbb{R}\ LIST\ SET\ SET$

$\forall n \bullet\ n\ Space = \{v \mid \#v = n\} \lhd_T ListMetric\ D_R\ MetricTopology$

The n-cube is the subpace of n-space comprising vectors with coordinates in the closed interval $[0, 1]$.

SML

```
declare_postfix(400, "Cube");
```

HOL Constant

$\boldsymbol{\$Cube} : \mathbb{N} \to \mathbb{R}\ LIST\ SET\ SET$

$\forall n \bullet\ n\ Cube = \{v \mid Elems\ v \subseteq ClosedInterval\ 0.\ 1.\} \lhd_T n\ Space$

The open n-cube is the subpace of n-space comprising vectors with coordinates in the open interval $(0, 1)$.

SML

```
declare_postfix(400, "OpenCube");
```

HOL Constant

$\boldsymbol{\$OpenCube} : \mathbb{N} \to \mathbb{R}\ LIST\ SET\ SET$

$\forall n \bullet\ n\ OpenCube = \{v \mid Elems\ v \subseteq OpenInterval\ 0.\ 1.\} \lhd_T n\ Space$

The (topological) n-sphere is the subpace of the n-cube comprising vectors with at least one coordinate in the set $\{0, 1\}$.

SML

```
declare_postfix(400, "Sphere");
```

HOL Constant

$\boldsymbol{\$Sphere} : \mathbb{N} \to \mathbb{R}\ LIST\ SET\ SET$

$\forall n \bullet\ n\ Sphere = \{v \mid \neg Elems\ v \cap \{0.; 1.\} = \{\}\} \lhd_T n\ Cube$

4 PATHS AND HOMOTOPY— DEFINITIONS

SML

```
force_delete_theory"homotopy" handle Fail _ => ();
open_theory"topology_ℝ";
new_theory"homotopy";
set_merge_pcs["basic_hol1", "'sets_alg", "'ℤ", "'ℝ"];
```

For convenience, we represent paths in a space as continuous functions on the whole real line. For the time being we do not define a topology on the path space (this was historically a slightly thorny topic in the literature and the "modern" solution via k-ification seems out of place at this stage).

SML

HOL Constant

$$\textbf{Paths} : 'a\ SET\ SET \to (\mathbb{R} \to 'a)\ SET$$

$$\begin{array}{lll} \forall \tau \bullet & Paths\ \tau = & \\ & \{ & f \\ & | & f \in (O_R,\ \tau)\ Continuous \\ & \wedge & (\forall x \bullet\ x \leq 0. \Rightarrow f\ x = f\ 0.) \\ & \wedge & (\forall x \bullet\ 1. \leq x \Rightarrow f\ x = f\ 1.)\} \end{array}$$

We now consider path connectedness. Here is the definition of a path connected set.

SML

```
declare_postfix(400, "PathConnected");
```

HOL Constant

$$\textbf{\$PathConnected} : 'a\ SET\ SET \to 'a\ SET\ SET$$

$$\begin{array}{llll} \forall \tau \bullet\ \tau\ PathConnected = & & & \\ & \{ & A & \\ & | & A \subseteq Space_T\ \tau & \\ & \wedge & \forall x\ y \bullet\ x \in A \wedge y \in A & \\ & \Rightarrow & \exists f \bullet & f \in Paths\ \tau \\ & & \wedge & (\forall\ t \bullet\ f\ t \in A) \\ & & \wedge & f\ (\mathbb{N}\mathbb{R}\ 0) = x \\ & & \wedge & f\ (\mathbb{N}\mathbb{R}\ 1) = y\} \end{array}$$

SML

HOL Constant

$$\textbf{LocallyPathConnected} : 'a\ SET\ SET\ SET$$

$$\begin{array}{ll} \forall \tau \bullet & \tau \in LocallyPathConnected \\ \Leftrightarrow & \forall x\ A \bullet x \in A \wedge A \in \tau \Rightarrow \exists B \bullet B \in \tau \wedge x \in B \wedge B \subseteq A \wedge B \in \tau\ PathConnected \end{array}$$

Continuing along the way towards the elements of algebraic topology, we now consider the notion of a homotopy. Here and elsewhere it is convenient to model functions continuous on the unit interval by functions continuous on the whole line. This is not problematic since any function continuous on the unit interval can be extended to be continuous everywhere.

Our homotopies are relative to a set X.

SML

```
declare_postfix(400, "Homotopy");
```

HOL Constant

$$\mathbf{\$Homotopy} : {}'a\ SET\ SET \times {}'a\ SET \times {}'b\ SET\ SET \to ({}'a \times \mathbb{R} \to {}'b)\ SET$$

$$\forall \sigma\ X\ \tau \bullet\ (\sigma,\ X,\ \tau)\ Homotopy = \{\ f \mid f \in ((\sigma \times_T O_R),\ \tau)\ Continuous \land \forall x\ s\ t \bullet x \in X \Rightarrow f(x,\ s) = f(x,\ t)\}$$

SML

```
declare_postfix(400, "HomotopyClass");
```

HOL Constant

$$\mathbf{\$HomotopyClass} : {}'a\ SET\ SET \times {}'a\ SET \times {}'b\ SET\ SET \to ({}'a \to {}'b) \to ({}'a \to {}'b)\ SET$$

$$\forall \sigma\ X\ \tau\ f \bullet\ ((\sigma,\ X,\ \tau)\ HomotopyClass)\ f = \{g \mid \exists H \bullet\ H \in (\sigma,\ X,\ \tau)\ Homotopy \land (\forall x \bullet\ H(x,\ \mathbb{NR}\ 0) = f\ x) \land (\forall x \bullet\ H(x,\ \mathbb{NR}\ 1) = g\ x)\}$$

4.1 The Path Groupoid

Now we define addition of paths:

SML

```
declare_infix(300, "+P");
```

HOL Constant

$$\mathbf{\$+_P} : (\mathbb{R} \to {}'a) \to (\mathbb{R} \to {}'a) \to (\mathbb{R} \to {}'a)$$

$$\forall f\ g \bullet\ f +_P g = (\lambda t \bullet if\ t \le 1/2\ then\ f\ (\mathbb{NR}\ 2*t)\ else\ g\ (\mathbb{NR}\ 2*(t - 1/2)))$$

The identity elements of the path space may be taken to be the constant paths of zero length:

HOL Constant

$$\mathbf{0_P} : {}'a \to (\mathbb{R} \to {}'a)$$

$$\forall x \bullet\ 0_P\ x = (\lambda t \bullet\ x)$$

Now we define the inverse of a path:

SML

HOL Constant

$$\mathbf{\$\sim_P} : (\mathbb{R} \to {}'a) \to (\mathbb{R} \to {}'a)$$

$$\forall f \bullet\ \sim_P f = (\lambda t \bullet\ f(\mathbb{NR}\ 1 -\ t))$$

We prove some basic facts about homotopies and paths.

path_connected_connected_thm
product_path_connected_thm
homotopy_class_refl_thm
homotopy_class_sym_thm
homotopy_class_trans_thm
homotopy_⊆_thm
homotopy_class_⊆_thm
homotopy_class_comp_left_thm
homotopy_class_comp_right_thm
homotopy_class_ℝ_thm
paths_continuous_thm
ppath_0_path_thm
path_plus_path_thm
path_minus_path_thm
path_plus_assoc_lemma1
path_plus_assoc_lemma2
path_plus_assoc_lemma3
path_plus_assoc_thm
path_plus_0_lemma1
path_plus_0_lemma2
path_plus_0_lemma3
path_plus_0_thm
path_0_plus_lemma1
path_0_plus_lemma2
path_0_plus_lemma3
path_0_plus_thm
path_plus_minus_lemma1
path_plus_minus_lemma2
path_plus_minus_lemma3
path_plus_minus_thm
path_minus_minus_thm
path_minus_plus_thm

We prove some facts about path connectedness and local path connectedness.

open_connected_path_connected_thm
open_interval_path_connected_thm
ℝ_locally_path_connected_thm
product_locally_path_connected_thm

We define the notion of homotopy lifting property for a pair comprising a topological space ρ and a continuous mapping p from a topological σ to a topological space τ as follows:

HOL Constant

HomotopyLiftingProperty :
    ('a SET SET × ('b → 'c) × 'b SET SET × 'c SET SET) SET

$\forall \rho\ \sigma\ \tau\ p \bullet$
    $(\rho, (p, \sigma, \tau)) \in HomotopyLiftingProperty$
$\Leftrightarrow$ $\rho \in Topology$
  $\wedge$ $\sigma \in Topology$
  $\wedge$ $\tau \in Topology$
  $\wedge$ $p \in (\sigma, \tau)\ Continuous$
  $\wedge$ $(\forall f\ h \bullet$
      $f \in (\rho, \sigma)\ Continuous$
    $\wedge$ $h \in (\rho \times_T O_R, \tau)\ Continuous$
    $\wedge$ $(\forall\ x \bullet\ x \in Space_T\ \rho \Rightarrow h\ (x,\ 0.) = p\ (f\ x))$
    $\Rightarrow$ $(\exists L \bullet$
        $L \in (\rho \times_T O_R, \sigma)\ Continuous$
      $\wedge$ $(\forall\ x \bullet\ x \in Space_T\ \rho \Rightarrow L\ (x,\ 0.) = f\ x)$
      $\wedge$ $(\forall\ x\ s \bullet$
          $x \in Space_T\ \rho$
        $\wedge$ $s \in ClosedInterval\ 0.\ 1.$

$$\Rightarrow \quad p\ (L\ (x,\ s)) = h\ (x,\ s))))$$

We prove that a covering project has the homotopy lifting property with respect to any space(i.e., it is a fibration) and hence the path-lifting property.

covering_projection_fibration_thm1
covering_projection_fibration_thm
covering_projection_path_lifting_thm

A THE THEORY topology

A.1 Parents

fincomb *bin_rel*

A.2 Children

metric_spaces

A.3 Constants

$\boldsymbol{Topology}$	$'a\ \mathbb{P}\ \mathbb{P}\ \mathbb{P}$
$\boldsymbol{Space_T}$	$'a\ \mathbb{P}\ \mathbb{P} \to {'a}\ \mathbb{P}$
$\$\boldsymbol{Closed}$	$'a\ \mathbb{P}\ \mathbb{P} \to {'a}\ \mathbb{P}\ \mathbb{P}$
$\$\lhd_{\boldsymbol{T}}$	$'a\ \mathbb{P} \to {'a}\ \mathbb{P}\ \mathbb{P} \to {'a}\ \mathbb{P}\ \mathbb{P}$
$\$\times_{\boldsymbol{T}}$	$'a\ \mathbb{P}\ \mathbb{P} \to {'b}\ \mathbb{P}\ \mathbb{P} \to ({'a} \leftrightarrow {'b})\ \mathbb{P}$
$\mathbf{1}_{\boldsymbol{T}}$	$ONE\ \mathbb{P}\ \mathbb{P}$
$\mathbf{\Pi}_{\boldsymbol{T}}$	$\mathbb{N} \to {'a}\ \mathbb{P}\ \mathbb{P} \to {'a}\ LIST\ \mathbb{P}\ \mathbb{P}$
$\$\boldsymbol{Continuous}$	$'a\ \mathbb{P}\ \mathbb{P} \times {'b}\ \mathbb{P}\ \mathbb{P} \to ({'a} \to {'b})\ \mathbb{P}$
$\boldsymbol{Hausdorff}$	$'a\ \mathbb{P}\ \mathbb{P}\ \mathbb{P}$
$\$\boldsymbol{Compact}$	$'a\ \mathbb{P}\ \mathbb{P} \to {'a}\ \mathbb{P}\ \mathbb{P}$
$\$\boldsymbol{Connected}$	$'a\ \mathbb{P}\ \mathbb{P} \to {'a}\ \mathbb{P}\ \mathbb{P}$
$\$\boldsymbol{Homeomorphism}$	$'a\ \mathbb{P}\ \mathbb{P} \times {'b}\ \mathbb{P}\ \mathbb{P} \to ({'a} \to {'b})\ \mathbb{P}$
$\$\boldsymbol{Closure}$	$'a\ \mathbb{P}\ \mathbb{P} \to {'a}\ \mathbb{P} \to {'a}\ \mathbb{P}$
$\$\boldsymbol{Boundary}$	$'a\ \mathbb{P}\ \mathbb{P} \to {'a}\ \mathbb{P} \to {'a}\ \mathbb{P}$
$\$\boldsymbol{Interior}$	$'a\ \mathbb{P}\ \mathbb{P} \to {'a}\ \mathbb{P} \to {'a}\ \mathbb{P}$
$\$\boldsymbol{CoveringProjection}$	$'a\ \mathbb{P}\ \mathbb{P} \times {'b}\ \mathbb{P}\ \mathbb{P} \to ({'a} \to {'b})\ \mathbb{P}$
$\boldsymbol{Space_K}$	$'a\ \mathbb{P} \leftrightarrow \mathbb{N} \to {'a}\ \mathbb{P}$
$\$\boldsymbol{Skeleton}$	$\mathbb{N} \to {'a}\ \mathbb{P} \leftrightarrow \mathbb{N} \to {'a}\ \mathbb{P}$
$\boldsymbol{Protocomplex}$	$'a\ \mathbb{P}\ \mathbb{P} \to ({'a}\ \mathbb{P} \leftrightarrow \mathbb{N})\ \mathbb{P}$

A.4 Fixity

Right Infix 280:

$\lhd_{\boldsymbol{T}}$

Right Infix 290:

$\times_{\boldsymbol{T}}$

Right Infix 400:

Boundary ***Closure*** ***Interior*** ***Skeleton***

Postfix 400: ***Closed*** ***Connected*** ***CoveringProjection*** ***Compact*** ***Continuous*** ***Homeomorphism***

A.5 Definitions

Topology $\vdash$ *Topology*

$$= \{\tau \mid (\forall\ V \bullet\ V \subseteq \tau \Rightarrow \bigcup V \in \tau) \wedge (\forall\ A\ B \bullet\ A \in \tau \wedge B \in \tau \Rightarrow A \cap B \in \tau)\}$$

Space$_T$ $\vdash \forall\ \tau \bullet\ Space_T\ \tau = \bigcup \tau$

Closed $\vdash \forall\ \tau \bullet\ \tau\ Closed = \{A \mid \exists\ B \bullet\ B \in \tau \wedge A = Space_T\ \tau \setminus B\}$

$\triangleleft_T$ $\vdash \forall\ X\ \tau \bullet\ X \triangleleft_T\ \tau = \{A \mid \exists\ B \bullet\ B \in \tau \wedge A = B \cap X\}$

$\times_T$ $\vdash \forall\ \sigma\ \tau$

$$\bullet\ \sigma \times_T \tau = \{C \mid \forall\ x\ y \bullet (x,\ y) \in C \Rightarrow (\exists\ A\ B \bullet\ A \in \sigma \wedge B \in \tau \wedge x \in A \wedge y \in B \wedge (A \times B) \subseteq C)\}$$

1_T $\vdash 1_T = \{\{\};\ \{One\}\}$

Π_T $\vdash \forall\ \tau\ n$

$$\bullet\ \Pi_T\ 0\ \tau = \{\{\};\ \{[]\}\} \wedge \Pi_T\ (n + 1)\ \tau = \{C \mid \neg\ [] \in C \wedge (\forall\ x\ v \bullet\ Cons\ x\ v \in C \Rightarrow (\exists\ A\ B \bullet\ A \in \tau \wedge B \in \Pi_T\ n\ \tau \wedge x \in A \wedge v \in B \wedge (\forall\ y\ w \bullet\ y \in A \wedge w \in B \Rightarrow Cons\ y\ w \in C)))\}$$

Continuous $\vdash \forall\ \sigma\ \tau$

$$\bullet\ (\sigma,\ \tau)\ Continuous = \{f \mid (\forall\ x \bullet\ x \in Space_T\ \sigma \Rightarrow f\ x \in Space_T\ \tau) \wedge (\forall\ A \bullet\ A \in \tau \Rightarrow \{x \mid x \in Space_T\ \sigma \wedge f\ x \in A\} \in \sigma)\}$$

Hausdorff $\vdash$ *Hausdorff*

$$= \{\tau \mid \forall\ x\ y \bullet\ x \in Space_T\ \tau \wedge y \in Space_T\ \tau \wedge \neg\ x = y \Rightarrow (\exists\ A\ B \bullet\ A \in \tau \wedge B \in \tau \wedge x \in A \wedge y \in B \wedge A \cap B = \{\})\}$$

Compact $\vdash \forall\ \tau$

$$
\begin{array}{l}
\bullet\ \tau\ \mathit{Compact} \\
\quad = \{A \\
\quad | A \subseteq \mathit{Space}_T\ \tau \\
\qquad \land (\forall\ V \\
\qquad \bullet\ V \subseteq \tau \land A \subseteq \bigcup V \\
\qquad\quad \Rightarrow (\exists\ W \bullet\ W \subseteq V \land W \in \mathit{Finite} \land A \subseteq \bigcup W))\}
\end{array}
$$

Connected $\vdash \forall\ \tau$

$$
\begin{array}{l}
\bullet\ \tau\ \mathit{Connected} \\
\quad = \{A \\
\quad | A \subseteq \mathit{Space}_T\ \tau \\
\qquad \land (\forall\ B\ C \\
\qquad \bullet\ B \in \tau \land C \in \tau \land A \subseteq B \cup C \land A \cap B \cap C = \{\} \\
\qquad\quad \Rightarrow A \subseteq B \lor A \subseteq C)\}
\end{array}
$$

Homeomorphism

$$
\begin{array}{l}
\vdash \forall\ \sigma\ \tau \\
\bullet\ (\sigma, \tau)\ \mathit{Homeomorphism} \\
\quad = \{f \\
\quad | f \in (\sigma, \tau)\ \mathit{Continuous} \\
\qquad \land (\exists\ g \\
\qquad \bullet\ g \in (\tau, \sigma)\ \mathit{Continuous} \\
\qquad\quad \land (\forall\ x \bullet\ x \in \mathit{Space}_T\ \sigma \Rightarrow g\ (f\ x) = x) \\
\qquad\quad \land (\forall\ y \bullet\ y \in \mathit{Space}_T\ \tau \Rightarrow f\ (g\ y) = y))\}
\end{array}
$$

Interior
Boundary
Closure $\vdash \forall\ \tau\ A$

$$
\begin{array}{l}
\bullet\ \tau\ \mathit{Interior}\ A = \{x | \exists\ B \bullet\ B \in \tau \land x \in B \land B \subseteq A\} \\
\quad \land\ \tau\ \mathit{Boundary}\ A \\
\qquad = \{x \\
\qquad | x \in \mathit{Space}_T\ \tau \\
\qquad\quad \land (\forall\ B \\
\qquad\quad \bullet\ B \in \tau \land x \in B \\
\qquad\qquad \Rightarrow \neg\ B \cap A = \{\} \land \neg\ B \setminus A = \{\})\} \\
\quad \land\ \tau\ \mathit{Closure}\ A \\
\qquad = \bigcap \{B | B \in \tau\ \mathit{Closed} \land A \cap \mathit{Space}_T\ \tau \subseteq B\}
\end{array}
$$

CoveringProjection

$$
\begin{array}{l}
\vdash \forall\ \sigma\ \tau \\
\bullet\ (\sigma, \tau)\ \mathit{CoveringProjection} \\
\quad = \{p \\
\quad | p \in (\sigma, \tau)\ \mathit{Continuous} \\
\qquad \land (\forall\ y \\
\qquad \bullet\ y \in \mathit{Space}_T\ \tau \\
\qquad\quad \Rightarrow (\exists\ C \\
\qquad\quad \bullet\ y \in C \\
\qquad\qquad \land\ C \in \tau \\
\qquad\qquad \land (\exists\ U \\
\qquad\qquad \bullet\ U \subseteq \sigma \\
\qquad\qquad\quad \land (\forall\ x \\
\qquad\qquad\quad \bullet\ x \in \mathit{Space}_T\ \sigma \land p\ x \in C \\
\qquad\qquad\qquad \Rightarrow (\exists\ A \bullet\ x \in A \land A \in U)) \\
\qquad\qquad\quad \land (\forall\ A\ B \\
\qquad\qquad\quad \bullet\ A \in U \land B \in U \land \neg\ A \cap B = \{\}
\end{array}
$$

$$\begin{array}{l}
\quad \Rightarrow A = B) \\
\land (\forall A \\
\bullet A \in U \\
\quad \Rightarrow p \\
\quad\ \in (A \lhd_T \sigma, \\
\qquad C \\
\qquad\quad \lhd_T \tau)\ \mathit{Homeomorphism})))) \}
\end{array}$$

$Space_K$ $\vdash \forall C \bullet Space_K\ C = \bigcup \{c | \exists m \bullet (c, m) \in C\}$

Skeleton $\vdash \forall n\ C \bullet n\ \mathit{Skeleton}\ C = \bigcup \{c | \exists m \bullet m \leq n \land (c, m) \in C\}$

Protocomplex
$$\begin{array}{l}
\vdash \forall C\ \tau \\
\quad \bullet C \in \mathit{Protocomplex}\ \tau \\
\qquad \Leftrightarrow (\forall c\ m \bullet (c, m) \in C \Rightarrow c \in \tau\ \mathit{Closed}) \\
\qquad\ \land (\forall x \\
\qquad\ \bullet x \in Space_K\ C \\
\qquad\quad \Rightarrow (\exists_1 (c, m) \\
\qquad\quad \bullet (c, m) \in C \\
\qquad\qquad \land x \in (m\ \mathit{Skeleton}\ C \lhd_T \tau)\ \mathit{Interior}\ c)) \\
\qquad\ \land (\forall A \\
\qquad\ \bullet A \subseteq Space_K\ C \\
\qquad\qquad \land (\forall c\ m \bullet (c, m) \in C \Rightarrow A \cap c \in \tau\ \mathit{Closed}) \\
\qquad\quad \Rightarrow A \in \tau\ \mathit{Closed}) \\
\qquad\ \land (\forall c\ m \\
\qquad\ \bullet (c, m) \in C \\
\qquad\quad \Rightarrow \{(d, n) \\
\qquad\qquad | (d, n) \in C \land n < m \land \lnot c \cap d = \{\}\} \\
\qquad\qquad \in \mathit{Finite})
\end{array}$$

A.6 Theorems

enum_set_⊆_thm
$\vdash \forall A\ B\ C \bullet \mathit{Insert}\ A\ B \subseteq C \Leftrightarrow A \in C \land B \subseteq C$

⋃_enum_set_clauses
$\vdash \bigcup \{\} = \{\} \land (\forall A\ B \bullet \bigcup (\mathit{Insert}\ A\ B) = A \cup \bigcup B)$

⋂_enum_set_clauses
$\vdash \bigcap \{\} = \mathit{Universe} \land (\forall A\ B \bullet \bigcap (\mathit{Insert}\ A\ B) = A \cap \bigcap B)$

finite_image_thm
$\vdash \forall f\ A \bullet A \in \mathit{Finite} \Rightarrow \{y | \exists x \bullet x \in A \land y = f\ x\} \in \mathit{Finite}$

⊆_size_thm $\vdash \forall a\ b \bullet a \in \mathit{Finite} \land b \subseteq a \Rightarrow \#\ b \leq \#\ a$

⊆_size_thm1 $\vdash \forall a\ b \bullet a \in \mathit{Finite} \land b \subseteq a \land \lnot b = a \Rightarrow \#\ b < \#\ a$

finite_⊆_well_founded_thm
$$\begin{array}{l}
\vdash \forall p\ a \\
\quad \bullet a \in \mathit{Finite} \land p\ a \\
\qquad \Rightarrow (\exists b \bullet b \subseteq a \land p\ b \land (\forall c \bullet c \subseteq b \land p\ c \Rightarrow c = b))
\end{array}$$

empty_open_thm
$\vdash \forall \tau \bullet \tau \in \mathit{Topology} \Rightarrow \{\} \in \tau$

space_t_open_thm
$\vdash \forall \tau \bullet \tau \in \mathit{Topology} \Rightarrow Space_T\ \tau \in \tau$

empty_closed_thm
$\vdash \forall \tau \bullet \tau \in \mathit{Topology} \Rightarrow \{\} \in \tau\ \mathit{Closed}$

space_t_closed_thm
$\vdash \forall \tau \bullet \tau \in \mathit{Topology} \Rightarrow Space_T\ \tau \in \tau\ \mathit{Closed}$

open_open_neighbourhood_thm

$\vdash \forall \tau\ A$
$\bullet\ \tau \in Topology$
$\Rightarrow (A \in \tau$
$\Leftrightarrow (\forall\ x \bullet\ x \in A \Rightarrow (\exists\ B \bullet\ B \in \tau \wedge x \in B \wedge B \subseteq A)))$

closed_open_neighbourhood_thm

$\vdash \forall \tau\ A$
$\bullet\ \tau \in Topology$
$\Rightarrow (A \in \tau\ Closed$
$\Leftrightarrow A \subseteq Space_T\ \tau$
$\wedge (\forall\ x$
$\bullet\ x \in Space_T\ \tau \wedge \neg\ x \in A$
$\Rightarrow (\exists\ B \bullet\ B \in \tau \wedge x \in B \wedge B \cap A = \{\})))$

∈_space_t_thm

$\vdash \forall \tau\ x\ A \bullet\ x \in A \wedge A \in \tau \Rightarrow x \in Space_T\ \tau$

∈_closed_∈_space_t_thm

$\vdash \forall \tau\ x\ A \bullet\ x \in A \wedge A \in \tau\ Closed \Rightarrow x \in Space_T\ \tau$

closed_open_complement_thm

$\vdash \forall \tau\ A$
$\bullet\ \tau \in Topology$
$\Rightarrow (A \in \tau\ Closed$
$\Leftrightarrow A \subseteq Space_T\ \tau \wedge Space_T\ \tau \setminus A \in \tau)$

∪_open_thm $\vdash \forall \tau\ A\ B \bullet\ \tau \in Topology \wedge A \in \tau \wedge B \in \tau \Rightarrow A \cup B \in \tau$

⋃_open_thm $\vdash \forall \tau\ V \bullet\ \tau \in Topology \wedge V \subseteq \tau \Rightarrow \bigcup\ V \in \tau$

∩_open_thm $\vdash \forall \tau\ A\ B \bullet\ \tau \in Topology \wedge A \in \tau \wedge B \in \tau \Rightarrow A \cap B \in \tau$

⋂_open_thm $\vdash \forall \tau\ V$
$\bullet\ \tau \in Topology \wedge \neg\ V = \{\} \wedge V \in Finite \wedge V \subseteq \tau$
$\Rightarrow \bigcap\ V \in \tau$

∩_closed_thm $\vdash \forall \tau\ A\ B$
$\bullet\ \tau \in Topology \wedge A \in \tau\ Closed \wedge B \in \tau\ Closed$
$\Rightarrow A \cap B \in \tau\ Closed$

⋂_closed_thm $\vdash \forall \tau\ V$
$\bullet\ \tau \in Topology \wedge \neg\ V = \{\} \wedge V \subseteq \tau\ Closed$
$\Rightarrow \bigcap\ V \in \tau\ Closed$

∪_closed_thm $\vdash \forall \tau\ A\ B$
$\bullet\ \tau \in Topology \wedge A \in \tau\ Closed \wedge B \in \tau\ Closed$
$\Rightarrow A \cup B \in \tau\ Closed$

⋃_closed_thm $\vdash \forall \tau\ V$
$\bullet\ \tau \in Topology \wedge \neg\ V = \{\} \wedge V \in Finite \wedge V \subseteq \tau\ Closed$
$\Rightarrow \bigcup\ V \in \tau\ Closed$

finite_⋂_open_thm

$\vdash \forall \tau\ V$
$\bullet\ \tau \in Topology \wedge V \subseteq \tau \wedge \neg\ V = \{\} \wedge V \in Finite$
$\Rightarrow \bigcap\ V \in \tau$

subspace_topology_thm

$\vdash \forall \tau\ X \bullet\ \tau \in Topology \Rightarrow X \lhd_T\ \tau \in Topology$

subspace_topology_space_t_thm

$\vdash \forall \tau\ A$
$\bullet\ \tau \in Topology \Rightarrow Space_T\ (A \lhd_T\ \tau) = A \cap Space_T\ \tau$

subspace_topology_space_t_thm1

$\vdash \forall \tau\ A$

$\bullet\ \tau \in Topology \land A \subseteq Space_T\ \tau$
$\Rightarrow Space_T\ (A \lhd_T \tau) = A$

universe_subspace_topology_thm
$\vdash \forall\ \tau \bullet\ Universe \lhd_T \tau = \tau$

open_⊆_space_t_thm
$\vdash \forall\ \tau\ A \bullet\ \tau \in Topology \land A \in \tau \Rightarrow A \subseteq Space_T\ \tau$

subspace_topology_space_t_thm2
$\vdash \forall\ \tau\ A \bullet\ \tau \in Topology \land A \in \tau \Rightarrow Space_T\ (A \lhd_T \tau) = A$

subspace_topology_space_t_thm3
$\vdash \forall\ \tau\ A$
$\bullet\ \tau \in Topology \land A \in \tau\ Closed \Rightarrow Space_T\ (A \lhd_T \tau) = A$

subspace_topology_closed_thm
$\vdash \forall\ X\ \tau$
$\bullet\ \tau \in Topology$
$\Rightarrow (X \lhd_T \tau)\ Closed$
$= \{A | \exists\ B \bullet\ B \in \tau\ Closed \land A = B \cap X\}$

trivial_subspace_topology_thm
$\vdash \forall\ \tau \bullet\ \tau \in Topology \Rightarrow Space_T\ \tau \lhd_T \tau = \tau$

⊆_subspace_topology_thm
$\vdash \forall\ \tau\ A\ B \bullet\ A \subseteq B \Rightarrow A \lhd_T B \lhd_T \tau = A \lhd_T \tau$

product_topology_thm
$\vdash \forall\ \sigma\ \tau$
$\bullet\ \sigma \in Topology \land \tau \in Topology \Rightarrow \sigma \times_T \tau \in Topology$

product_topology_space_t_thm
$\vdash \forall\ \sigma\ \tau$
$\bullet\ \sigma \in Topology \land \tau \in Topology$
$\Rightarrow Space_T\ (\sigma \times_T \tau) = (Space_T\ \sigma \times Space_T\ \tau)$

unit_topology_thm
$\vdash 1_T \in Topology$

space_t_unit_topology_thm
$\vdash Space_T\ 1_T = Universe$

power_topology_length_thm
$\vdash \forall\ \tau\ n\ v \bullet\ v \in Space_T\ (\Pi_T\ n\ \tau) \Rightarrow \#\ v = n$

power_topology_thm
$\vdash \forall\ \tau\ n \bullet\ \tau \in Topology \Rightarrow \Pi_T\ n\ \tau \in Topology$

continuous_∈_space_t_thm
$\vdash \forall\ \sigma\ \tau\ f\ x$
$\bullet\ f \in (\sigma, \tau)\ Continuous \land x \in Space_T\ \sigma$
$\Rightarrow f\ x \in Space_T\ \tau$

continuous_open_thm
$\vdash \forall\ \sigma\ \tau\ f\ A$
$\bullet\ f \in (\sigma, \tau)\ Continuous \land A \in \tau$
$\Rightarrow \{x | x \in Space_T\ \sigma \land f\ x \in A\} \in \sigma$

continuous_closed_thm
$\vdash \forall\ \sigma\ \tau$
$\bullet\ (\sigma, \tau)\ Continuous$
$= \{f$
$|(\forall\ x \bullet\ x \in Space_T\ \sigma \Rightarrow f\ x \in Space_T\ \tau)$
$\land (\forall\ A$
$\bullet\ A \in \tau\ Closed$
$\Rightarrow \{x | x \in Space_T\ \sigma \land f\ x \in A\}$

$$\in \sigma\ Closed)\}$$

subspace_continuous_thm

$$\vdash \forall\ \sigma\ \tau\ A\ B\ f \bullet \sigma \in Topology \land \tau \in Topology \land f \in (\sigma, \tau)\ Continuous \land (\forall\ x \bullet\ x \in A \Rightarrow f\ x \in B) \Rightarrow f \in (A \lhd_T \sigma, B \lhd_T \tau)\ Continuous$$

subspace_domain_continuous_thm

$$\vdash \forall\ \sigma\ \tau\ A\ B\ f \bullet \sigma \in Topology \land \tau \in Topology \land f \in (\sigma, \tau)\ Continuous \Rightarrow f \in (A \lhd_T \sigma, \tau)\ Continuous$$

empty_continuous_thm

$$\vdash \forall\ \sigma\ \tau\ f \bullet \sigma \in Topology \land \tau \in Topology \Rightarrow f \in (\{\} \lhd_T \sigma, \tau)\ Continuous$$

subspace_range_continuous_thm

$$\vdash \forall\ \sigma\ \tau\ f\ B \bullet \sigma \in Topology \land \tau \in Topology \land f \in (\sigma, B \lhd_T \tau)\ Continuous \Rightarrow f \in (\sigma, \tau)\ Continuous$$

subspace_range_continuous_⇔_thm

$$\vdash \forall\ \sigma\ \tau\ f\ B \bullet \sigma \in Topology \land \tau \in Topology \land B \subseteq Space_T\ \tau \Rightarrow (f \in (\sigma, B \lhd_T \tau)\ Continuous \Leftrightarrow f \in (\sigma, \tau)\ Continuous \land (\forall\ x \bullet\ x \in Space_T\ \sigma \Rightarrow f\ x \in B))$$

subspace_range_continuous_bc_thm

$$\vdash \forall\ \sigma\ \tau\ f\ B \bullet \sigma \in Topology \land \tau \in Topology \land B \subseteq Space_T\ \tau \land (\forall\ x \bullet\ x \in Space_T\ \sigma \Rightarrow f\ x \in B) \land f \in (\sigma, \tau)\ Continuous \Rightarrow f \in (\sigma, B \lhd_T \tau)\ Continuous$$

const_continuous_thm

$$\vdash \forall\ \sigma\ \tau\ c \bullet \sigma \in Topology \land \tau \in Topology \land c \in Space_T\ \tau \Rightarrow (\lambda\ x \bullet\ c) \in (\sigma, \tau)\ Continuous$$

id_continuous_thm

$$\vdash \forall\ \tau \bullet\ \tau \in Topology \Rightarrow (\lambda\ x \bullet\ x) \in (\tau, \tau)\ Continuous$$

comp_continuous_thm

$$\vdash \forall\ f\ g\ \rho\ \sigma\ \tau \bullet f \in (\rho, \sigma)\ Continuous \land g \in (\sigma, \tau)\ Continuous \land \rho \in Topology \land \sigma \in Topology \land \tau \in Topology \Rightarrow (\lambda\ x \bullet\ g\ (f\ x)) \in (\rho, \tau)\ Continuous$$

left_proj_continuous_thm

$\vdash \forall\ \sigma\ \tau$
$\bullet\ \sigma \in\ \mathit{Topology} \land \tau \in\ \mathit{Topology}$
$\Rightarrow (\lambda\ (x,\ y)\bullet\ x) \in (\sigma \times_T\ \tau,\ \sigma)\ \mathit{Continuous}$

fst_continuous_thm

$\vdash \forall\ \sigma\ \tau$
$\bullet\ \sigma \in\ \mathit{Topology} \land \tau \in\ \mathit{Topology}$
$\Rightarrow \mathit{Fst} \in (\sigma \times_T\ \tau,\ \sigma)\ \mathit{Continuous}$

right_proj_continuous_thm

$\vdash \forall\ \sigma\ \tau$
$\bullet\ \sigma \in\ \mathit{Topology} \land \tau \in\ \mathit{Topology}$
$\Rightarrow (\lambda\ (x,\ y)\bullet\ y) \in (\sigma \times_T\ \tau,\ \tau)\ \mathit{Continuous}$

snd_continuous_thm

$\vdash \forall\ \sigma\ \tau$
$\bullet\ \sigma \in\ \mathit{Topology} \land \tau \in\ \mathit{Topology}$
$\Rightarrow \mathit{Snd} \in (\sigma \times_T\ \tau,\ \tau)\ \mathit{Continuous}$

product_continuous_thm

$\vdash \forall\ f\ g\ \rho\ \sigma\ \tau$
$\bullet\ f \in (\rho,\ \sigma)\ \mathit{Continuous}$
$\land\ g \in (\rho,\ \tau)\ \mathit{Continuous}$
$\land\ \rho \in\ \mathit{Topology}$
$\land\ \sigma \in\ \mathit{Topology}$
$\land\ \tau \in\ \mathit{Topology}$
$\Rightarrow (\lambda\ z\bullet\ (f\ z,\ g\ z)) \in (\rho,\ \sigma \times_T\ \tau)\ \mathit{Continuous}$

product_continuous_⇔_thm

$\vdash \forall\ f\ g\ \rho\ \sigma\ \tau$
$\bullet\ \rho \in\ \mathit{Topology} \land \sigma \in\ \mathit{Topology} \land \tau \in\ \mathit{Topology}$
$\Rightarrow ((\lambda\ z\bullet\ (f\ z,\ g\ z)) \in (\rho,\ \sigma \times_T\ \tau)\ \mathit{Continuous}$
$\Leftrightarrow f \in (\rho,\ \sigma)\ \mathit{Continuous}$
$\land\ g \in (\rho,\ \tau)\ \mathit{Continuous})$

left_product_inj_continuous_thm

$\vdash \forall\ \sigma\ \tau\ y$
$\bullet\ \sigma \in\ \mathit{Topology} \land \tau \in\ \mathit{Topology} \land y \in\ \mathit{Space}_T\ \tau$
$\Rightarrow (\lambda\ x\bullet\ (x,\ y)) \in (\sigma,\ \sigma \times_T\ \tau)\ \mathit{Continuous}$

right_product_inj_continuous_thm

$\vdash \forall\ \sigma\ \tau\ x$
$\bullet\ \sigma \in\ \mathit{Topology} \land \tau \in\ \mathit{Topology} \land x \in\ \mathit{Space}_T\ \sigma$
$\Rightarrow (\lambda\ y\bullet\ (x,\ y)) \in (\tau,\ \sigma \times_T\ \tau)\ \mathit{Continuous}$

range_unit_topology_continuous_thm

$\vdash \forall\ \tau\ f\bullet\ \tau \in\ \mathit{Topology} \Rightarrow f \in (\tau,\ \mathit{1}_T)\ \mathit{Continuous}$

domain_unit_topology_continuous_thm

$\vdash \forall\ \tau\ f$
$\bullet\ \tau \in\ \mathit{Topology} \land f\ \mathit{One} \in\ \mathit{Space}_T\ \tau$
$\Rightarrow f \in (\mathit{1}_T,\ \tau)\ \mathit{Continuous}$

diag_inj_continuous_thm

$\vdash \forall\ \tau$
$\bullet\ \tau \in\ \mathit{Topology}$
$\Rightarrow (\lambda\ x\bullet\ (x,\ x)) \in (\tau,\ \tau \times_T\ \tau)\ \mathit{Continuous}$

cond_continuous_thm

$\vdash \forall\ f\ g\ X\ \sigma\ \tau$
$\bullet\ f \in (\sigma,\ \tau)\ \mathit{Continuous}$
$\land\ g \in (\sigma,\ \tau)\ \mathit{Continuous}$

$\wedge$ ($\forall$ x
$\bullet$ $x \in Space_T\ \sigma$
$\wedge$ ($\forall$ A
$\bullet$ $x \in A \wedge A \in \sigma$
$\Rightarrow$ ($\exists$ y z
$\bullet$ $y \in A \wedge z \in A \wedge y \in X \wedge \neg\ z \in X$))
$\Rightarrow f\ x = g\ x$)
$\wedge$ $\sigma \in Topology$
$\wedge$ $\tau \in Topology$
$\Rightarrow$ (λ $x\bullet$ *if* $x \in X$ *then* $f\ x$ *else* $g\ x$)
$\in (\sigma, \tau)$ *Continuous*

closed_∪_closed_continuous_thm

$\vdash \forall\ \sigma\ \tau\ A\ B\ f\ g$
$\bullet$ $\sigma \in Topology$
$\wedge$ $\tau \in Topology$
$\wedge$ $A \in \sigma$ *Closed*
$\wedge$ $B \in \sigma$ *Closed*
$\wedge$ $f \in (A \lhd_T \sigma, \tau)$ *Continuous*
$\wedge$ $g \in (B \lhd_T \sigma, \tau)$ *Continuous*
$\wedge$ ($\forall$ $x\bullet$ $x \in A \cap B \Rightarrow f\ x = g\ x$)
$\Rightarrow$ (λ $x\bullet$ *if* $x \in A$ *then* $f\ x$ *else* $g\ x$)
$\in ((A \cup B) \lhd_T \sigma, \tau)$ *Continuous*

open_∪_open_continuous_thm

$\vdash \forall\ \sigma\ \tau\ A\ B\ f\ g$
$\bullet$ $\sigma \in Topology$
$\wedge$ $\tau \in Topology$
$\wedge$ $A \in \sigma$
$\wedge$ $B \in \sigma$
$\wedge$ $f \in (A \lhd_T \sigma, \tau)$ *Continuous*
$\wedge$ $g \in (B \lhd_T \sigma, \tau)$ *Continuous*
$\wedge$ ($\forall$ $x\bullet$ $x \in A \cap B \Rightarrow f\ x = g\ x$)
$\Rightarrow$ (λ $x\bullet$ *if* $x \in A$ *then* $f\ x$ *else* $g\ x$)
$\in ((A \cup B) \lhd_T \sigma, \tau)$ *Continuous*

compatible_family_continuous_thm

$\vdash \forall\ \sigma\ \tau\ X\ U\ G$
$\bullet$ $\sigma \in Topology$
$\wedge$ $\tau \in Topology$
$\wedge$ ($\forall$ $x\bullet$ $x \in X \Rightarrow U\ x \subseteq X$)
$\wedge$ ($\forall$ $x\bullet$ $x \in X \Rightarrow x \in U\ x$)
$\wedge$ ($\forall$ $x\bullet$ $x \in X \Rightarrow U\ x \in X \lhd_T \sigma$)
$\wedge$ ($\forall$ x
$\bullet$ $x \in X \Rightarrow G\ x \in (U\ x \lhd_T \sigma, \tau)$ *Continuous*)
$\wedge$ ($\forall$ x $y\bullet$ $x \in X \wedge y \in U\ x \Rightarrow G\ y\ y = G\ x\ y$)
$\Rightarrow$ (λ $x\bullet$ $G\ x\ x$) $\in (X \lhd_T \sigma, \tau)$ *Continuous*

compatible_family_continuous_thm1

$\vdash \forall\ \sigma\ \tau\ X\ U\ G$
$\bullet$ $\sigma \in Topology$
$\wedge$ $\tau \in Topology$
$\wedge$ ($\forall$ v $r\bullet$ $(v, r) \in X \Rightarrow U\ (v, r) \subseteq X$)
$\wedge$ ($\forall$ v $r\bullet$ $(v, r) \in X \Rightarrow (v, r) \in U\ (v, r)$)
$\wedge$ ($\forall$ v $r\bullet$ $(v, r) \in X \Rightarrow U\ (v, r) \in X \lhd_T \sigma$)

$\land\ (\forall\ v\ r$

$\bullet\ (v,\ r) \in X$

$\Rightarrow\ G\ (v,\ r)$

$\in (U\ (v,\ r) \lhd_T \sigma,\ \tau)\ \mathit{Continuous})$

$\land\ (\forall\ v\ r\ w\ s$

$\bullet\ (v,\ r) \in X \land (w,\ s) \in U\ (v,\ r)$

$\Rightarrow\ G\ (w,\ s)\ (w,\ s) = G\ (v,\ r)\ (w,\ s))$

$\Rightarrow (\lambda\ (v,\ r) \bullet\ G\ (v,\ r)\ (v,\ r))$

$\in (X \lhd_T \sigma,\ \tau)\ \mathit{Continuous}$

same_on_space_continuous_thm

$\vdash \forall\ \sigma\ \tau\ f\ g$

$\bullet\ \sigma \in \mathit{Topology}$

$\land\ \tau \in \mathit{Topology}$

$\land\ g \in (\sigma,\ \tau)\ \mathit{Continuous}$

$\land\ (\forall\ x \bullet\ x \in \mathit{Space}_T\ \sigma \Rightarrow f\ x = g\ x)$

$\Rightarrow f \in (\sigma,\ \tau)\ \mathit{Continuous}$

same_on_space_continuous_thm1

$\vdash \forall\ \sigma\ \tau\ f\ g$

$\bullet\ \sigma \in \mathit{Topology}$

$\land\ \tau \in \mathit{Topology}$

$\land\ (\forall\ x \bullet\ x \in \mathit{Space}_T\ \sigma \Rightarrow f\ x = g\ x)$

$\Rightarrow (f \in (\sigma,\ \tau)\ \mathit{Continuous} \Leftrightarrow g \in (\sigma,\ \tau)\ \mathit{Continuous})$

subspace_product_continuous_thm

$\vdash \forall\ \rho\ \sigma\ \tau\ f\ A\ B$

$\bullet\ \rho \in \mathit{Topology}$

$\land\ \sigma \in \mathit{Topology}$

$\land\ \tau \in \mathit{Topology}$

$\land\ \neg\ (A \times B) = \{\}$

$\land\ A \subseteq \mathit{Space}_T\ \rho$

$\land\ B \subseteq \mathit{Space}_T\ \sigma$

$\Rightarrow (f \in ((A \times B) \lhd_T \rho \times_T \sigma,\ \tau)\ \mathit{Continuous}$

$\Leftrightarrow (\forall\ a\ b \bullet\ a \in A \land b \in B \Rightarrow f\ (a,\ b) \in \mathit{Space}_T\ \tau)$

$\land\ (\forall\ a\ b\ E$

$\bullet\ a \in A \land b \in B \land f\ (a,\ b) \in E \land E \in \tau$

$\Rightarrow (\exists\ C\ D$

$\bullet\ a \in C$

$\land\ C \in \rho$

$\land\ b \in D$

$\land\ D \in \sigma$

$\land\ (\forall\ x\ y$

$\bullet\ x \in A \cap C \land y \in B \cap D$

$\Rightarrow f\ (x,\ y) \in E))))$

subspace_topology_hausdorff_thm

$\vdash \forall\ \tau\ X$

$\bullet\ \tau \in \mathit{Topology} \land \tau \in \mathit{Hausdorff} \Rightarrow X \lhd_T \tau \in \mathit{Hausdorff}$

product_topology_hausdorff_thm

$\vdash \forall\ \sigma\ \tau$

$\bullet\ \sigma \in \mathit{Topology}$

$\land\ \tau \in \mathit{Topology}$

$\land\ \sigma \in \mathit{Hausdorff}$

$\land\ \tau \in \mathit{Hausdorff}$

$\Rightarrow \sigma \times_T \tau \in Hausdorff$

punctured_hausdorff_thm

$\vdash \forall \tau\ X\ x$
$\bullet\ \tau \in Topology$
$\land\ \tau \in Hausdorff$
$\land\ X \subseteq Space_T\ \tau$
$\land\ x \in Space_T\ \tau$
$\Rightarrow X \setminus \{x\} \in X \lhd_T \tau$

compact_topological_thm

$\vdash \forall \tau\ X$
$\bullet\ \tau \in Topology$
$\Rightarrow (X \in \tau\ Compact \Leftrightarrow X \in (X \lhd_T \tau)\ Compact)$

image_compact_thm

$\vdash \forall f\ C\ \sigma\ \tau$
$\bullet\ f \in (\sigma, \tau)\ Continuous$
$\land\ C \in \sigma\ Compact$
$\land\ \sigma \in Topology$
$\land\ \tau \in Topology$
$\Rightarrow \{y | \exists\ x \bullet\ x \in C \land y = f\ x\} \in \tau\ Compact$

$\cup$***_compact_thm***

$\vdash \forall C\ D\ \sigma$
$\bullet\ C \in \sigma\ Compact \land D \in \sigma\ Compact \land \sigma \in Topology$
$\Rightarrow C \cup D \in \sigma\ Compact$

compact_closed_thm

$\vdash \forall \tau\ C$
$\bullet\ \tau \in Topology \land \tau \in Hausdorff \land C \in \tau\ Compact$
$\Rightarrow C \in \tau\ Closed$

closed_$\subseteq$***_compact_thm***

$\vdash \forall \tau\ B\ C$
$\bullet\ \tau \in Topology$
$\land\ \tau \in Hausdorff$
$\land\ C \in \tau\ Compact$
$\land\ B \in \tau\ Closed$
$\land\ B \subseteq C$
$\Rightarrow B \in \tau\ Compact$

compact_basis_thm

$\vdash \forall U\ \tau\ X$
$\bullet\ \tau \in Topology$
$\land\ U \subseteq \tau$
$\land\ (\forall A\ x$
$\bullet\ x \in A \land A \in \tau \Rightarrow (\exists\ B \bullet\ x \in B \land B \subseteq A \land B \in U))$
$\land\ X \subseteq Space_T\ \tau$
$\land\ (\forall V$
$\bullet\ V \subseteq U \land X \subseteq \bigcup V$
$\Rightarrow (\exists\ W \bullet\ W \subseteq V \land W \in Finite \land X \subseteq \bigcup W))$
$\Rightarrow X \in \tau\ Compact$

compact_basis_product_topology_thm

$\vdash \forall \sigma\ \tau\ X$
$\bullet\ \sigma \in Topology$
$\land\ \tau \in Topology$
$\land\ X \subseteq Space_T\ (\sigma \times_T \tau)$

$\wedge (\forall\ V$
$\bullet\ V \subseteq \sigma \times_T \tau$
$\wedge (\forall\ D$
$\bullet\ D \in V$
$\Rightarrow (\exists\ B\ C$
$\bullet\ B \in \sigma \wedge C \in \tau \wedge D = (B \times C)))$
$\wedge X \subseteq \bigcup V$
$\Rightarrow (\exists\ W \bullet\ W \subseteq V \wedge W \in Finite \wedge X \subseteq \bigcup W))$
$\Rightarrow X \in (\sigma \times_T \tau)\ Compact$

product_compact_thm

$\vdash \forall\ X\ Y\ \sigma\ \tau$
$\bullet\ X \in \sigma\ Compact$
$\wedge Y \in \tau\ Compact$
$\wedge \sigma \in Topology$
$\wedge \tau \in Topology$
$\Rightarrow (X \times Y) \in (\sigma \times_T \tau)\ Compact$

compact_sequentially_compact_thm

$\vdash \forall\ \tau\ X\ s$
$\bullet\ \tau \in Topology \wedge X \in \tau\ Compact \wedge (\forall\ m \bullet\ s\ m \in X)$
$\Rightarrow (\exists\ x$
$\bullet\ x \in X$
$\wedge (\forall\ A$
$\bullet\ A \in \tau \wedge x \in A$
$\Rightarrow (\forall\ m \bullet\ \exists\ n \bullet\ m \le n \wedge s\ n \in A)))$

connected_topological_thm

$\vdash \forall\ \tau\ X$
$\bullet\ \tau \in Topology$
$\Rightarrow (X \in \tau\ Connected \Leftrightarrow X \in (X \lhd_T \tau)\ Connected)$

connected_closed_thm

$\vdash \forall\ \tau\ X$
$\bullet\ \tau\ Connected$
$= \{A$
$| A \subseteq Space_T\ \tau$
$\wedge (\forall\ B\ C$
$\bullet\ B \in \tau\ Closed$
$\wedge C \in \tau\ Closed$
$\wedge A \subseteq B \cup C$
$\wedge A \cap B \cap C = \{\}$
$\Rightarrow A \subseteq B \vee A \subseteq C)\}$

connected_pointwise_thm

$\vdash \forall\ \tau\ X$
$\bullet\ \tau \in Topology$
$\Rightarrow (X \in \tau\ Connected$
$\Leftrightarrow (\forall\ x\ y$
$\bullet\ x \in X \wedge y \in X$
$\Rightarrow (\exists\ Y$
$\bullet\ Y \subseteq X$
$\wedge x \in Y$
$\wedge y \in Y$
$\wedge Y \in \tau\ Connected)))$

connected_pointwise_bc_thm

$\vdash \forall\ \tau\ X$
$\bullet\ \tau \in Topology$
$\wedge\ (\forall\ x\ y$
$\bullet\ x \in X \wedge y \in X$
$\Rightarrow (\exists\ Y$
$\bullet\ Y \subseteq X \wedge x \in Y \wedge y \in Y \wedge Y \in \tau\ Connected))$
$\Rightarrow X \in \tau\ Connected$

empty_connected_thm

$\vdash \forall\ \tau\bullet\ \tau \in Topology \Rightarrow \{\} \in \tau\ Connected$

singleton_connected_thm

$\vdash \forall\ \tau\ x$
$\bullet\ \tau \in Topology \wedge x \in Space_T\ \tau \Rightarrow \{x\} \in \tau\ Connected$

image_connected_thm

$\vdash \forall\ f\ X\ \sigma\ \tau$
$\bullet\ f \in (\sigma, \tau)\ Continuous$
$\wedge\ X \in \sigma\ Connected$
$\wedge\ \sigma \in Topology$
$\wedge\ \tau \in Topology$
$\Rightarrow \{y | \exists\ x\bullet\ x \in X \wedge y = f\ x\} \in \tau\ Connected$

$\cup$***_connected_thm***

$\vdash \forall\ C\ D\ \sigma$
$\bullet\ \sigma \in Topology$
$\wedge\ C \in \sigma\ Connected$
$\wedge\ D \in \sigma\ Connected$
$\wedge\ \neg\ C \cap D = \{\}$
$\Rightarrow C \cup D \in \sigma\ Connected$

product_connected_thm

$\vdash \forall\ X\ Y\ \sigma\ \tau$
$\bullet\ X \in \sigma\ Connected$
$\wedge\ Y \in \tau\ Connected$
$\wedge\ \sigma \in Topology$
$\wedge\ \tau \in Topology$
$\Rightarrow (X \times Y) \in (\sigma \times_T \tau)\ Connected$

$\cup$***_open_connected_thm***

$\vdash \forall\ A\ B\ \sigma$
$\bullet\ A \in \sigma$
$\wedge\ \neg\ A = \{\}$
$\wedge\ B \in \sigma$
$\wedge\ \neg\ B = \{\}$
$\wedge\ A \cup B \in \sigma\ Connected$
$\Rightarrow \neg\ A \cap B = \{\}$

$\cup$***_closed_connected_thm***

$\vdash \forall\ A\ B\ \sigma$
$\bullet\ A \in \sigma\ Closed$
$\wedge\ \neg\ A = \{\}$
$\wedge\ B \in \sigma\ Closed$
$\wedge\ \neg\ B = \{\}$
$\wedge\ A \cup B \in \sigma\ Connected$
$\Rightarrow \neg\ A \cap B = \{\}$

$\cup$_$\cup$***_connected_thm***

$\vdash \forall\ C\ D\ E\ \sigma$

$\bullet\ \sigma \in \mathit{Topology}$
$\quad \land\ C \in \sigma\ \mathit{Connected}$
$\quad \land\ D \in \sigma\ \mathit{Connected}$
$\quad \land\ E \in \sigma\ \mathit{Connected}$
$\quad \land\ \neg\ C \cap D = \{\}$
$\quad \land\ \neg\ D \cap E = \{\}$
$\quad \Rightarrow\ C \cup D \cup E \in \sigma\ \mathit{Connected}$

cover_connected_thm

$\vdash \forall\ C\ U\ \sigma$
$\bullet\ \sigma \in \mathit{Topology}$
$\quad \land\ C \in \sigma\ \mathit{Connected}$
$\quad \land\ U \subseteq \sigma\ \mathit{Connected}$
$\quad \land\ C \subseteq \bigcup\ U$
$\quad \Rightarrow\ \bigcup\ \{D | D \in U\ \land\ \neg\ C \cap D = \{\}\} \in \sigma\ \mathit{Connected}$

separation_thm

$\vdash \forall\ \tau\ C\ D$
$\bullet\ \tau \in \mathit{Topology}$
$\quad \land\ C \in \tau\ \mathit{Connected}$
$\quad \land\ D \in \tau\ \mathit{Connected}$
$\quad \land\ \neg\ C \cup D \in \tau\ \mathit{Connected}$
$\quad \Rightarrow\ (\exists\ A\ B$
$\quad \bullet\ A \in \tau$
$\quad\quad \land\ B \in \tau$
$\quad\quad \land\ (C \cup D) \cap A \cap B = \{\}$
$\quad\quad \land\ C \subseteq A$
$\quad\quad \land\ D \subseteq B)$

finite_separation_thm

$\vdash \forall\ \tau\ U\ A$
$\bullet\ \tau \in \mathit{Topology}$
$\quad \land\ U \in \mathit{Finite}$
$\quad \land\ \neg\ \{\} \in U$
$\quad \land\ U \subseteq \tau\ \mathit{Connected}$
$\quad \land\ A \in U$
$\quad \land\ (\forall\ B$
$\quad \bullet\ B \in U\ \land\ \neg\ A = B \Rightarrow \neg\ A \cup B \in \tau\ \mathit{Connected})$
$\quad \Rightarrow\ (\exists\ C\ D$
$\quad \bullet\ C \in \tau$
$\quad\quad \land\ D \in \tau$
$\quad\quad \land\ A \subseteq C$
$\quad\quad \land\ \bigcup\ (U \setminus \{A\}) \subseteq D$
$\quad\quad \land\ \bigcup\ U \cap C \cap D = \{\})$

connected_extension_thm

$\vdash \forall\ \tau\ U\ B$
$\bullet\ \tau \in \mathit{Topology}$
$\quad \land\ U \in \mathit{Finite}$
$\quad \land\ \neg\ \{\} \in U$
$\quad \land\ U \subseteq \tau\ \mathit{Connected}$
$\quad \land\ B \in \tau\ \mathit{Connected}$
$\quad \land\ \bigcup\ U \cup B \in \tau\ \mathit{Connected}$
$\quad \land\ \neg\ \bigcup\ U \subseteq B$
$\quad \Rightarrow\ (\exists\ A \bullet\ A \in U\ \land\ A \cup B \in \tau\ \mathit{Connected}\ \land\ \neg\ A \subseteq B)$

connected_chain_thm

$$\vdash \forall\ \tau\ U\ A$$
$$\bullet\ \tau \in \mathit{Topology}$$
$$\land\ U \in \mathit{Finite}$$
$$\land\ \neg\ \{\} \in U$$
$$\land\ U \subseteq \tau\ \mathit{Connected}$$
$$\land\ \bigcup\ U \in \tau\ \mathit{Connected}$$
$$\land\ A \in U$$
$$\Rightarrow (\exists\ L\ n$$
$$\bullet\ L\ \mathit{0} = [A]$$
$$\land\ (\forall\ m\bullet \bigcup\ (\mathit{Elems}\ (L\ m)) \in \tau\ \mathit{Connected})$$
$$\land\ (\forall\ m\bullet\ \mathit{Elems}\ (L\ m) \subseteq U)$$
$$\land\ (\forall\ m$$
$$\bullet\ m < n$$
$$\Rightarrow (\exists\ B$$
$$\bullet\ B \in U$$
$$\land\ \neg\ B \subseteq \bigcup\ (\mathit{Elems}\ (L\ m))$$
$$\land\ L\ (m + \mathit{1}) = \mathit{Cons}\ B\ (L\ m)))$$
$$\land\ \bigcup\ U = \bigcup\ (\mathit{Elems}\ (L\ n))$$
$$\land\ (\forall\ m\bullet\ L\ m \in \mathit{Distinct}))$$

connected_triad_thm

$$\vdash \forall\ \tau\ A\ B\ C$$
$$\bullet\ \tau \in \mathit{Topology}$$
$$\land\ A \in \tau\ \mathit{Connected}$$
$$\land\ B \in \tau\ \mathit{Connected}$$
$$\land\ C \in \tau\ \mathit{Connected}$$
$$\land\ A \cup B \cup C \in \tau\ \mathit{Connected}$$
$$\Rightarrow A \cup C \in \tau\ \mathit{Connected} \lor B \cup C \in \tau\ \mathit{Connected}$$

connected_step_thm

$$\vdash \forall\ \tau\ U\ A$$
$$\bullet\ \tau \in \mathit{Topology}$$
$$\land\ U \in \mathit{Finite}$$
$$\land\ U \subseteq \tau\ \mathit{Connected}$$
$$\land\ \bigcup\ U \in \tau\ \mathit{Connected}$$
$$\land\ A \in U$$
$$\Rightarrow A = \bigcup\ U$$
$$\lor\ (\exists\ B\ V$$
$$\bullet\ B \in U$$
$$\land\ \neg\ B = A$$
$$\land\ V \subseteq U$$
$$\land\ \bigcup\ V \in \tau\ \mathit{Connected}$$
$$\land\ \neg\ B \subseteq \bigcup\ V$$
$$\land\ \bigcup\ U = B \cup \bigcup\ V)$$

id_homomorphism_thm

$$\vdash \forall\ \tau\bullet\ \tau \in \mathit{Topology} \Rightarrow (\lambda\ x\bullet\ x) \in (\tau, \tau)\ \mathit{Homeomorphism}$$

comp_homeomorphism_thm

$$\vdash \forall\ f\ g\ \rho\ \sigma\ \tau$$
$$\bullet\ f \in (\rho, \sigma)\ \mathit{Homeomorphism}$$
$$\land\ g \in (\sigma, \tau)\ \mathit{Homeomorphism}$$
$$\land\ \rho \in \mathit{Topology}$$
$$\land\ \sigma \in \mathit{Topology}$$

$\wedge\ \tau \in \mathit{Topology}$

$\Rightarrow (\lambda\ x\bullet\ g\ (f\ x)) \in (\rho,\ \tau)\ \mathit{Homeomorphism}$

product_homeomorphism_thm

$\vdash \forall\ f\ g\ \rho\ \sigma\ \tau\ \upsilon$

$\bullet\ f \in (\rho,\ \sigma)\ \mathit{Homeomorphism}$

$\wedge\ g \in (\tau,\ \upsilon)\ \mathit{Homeomorphism}$

$\wedge\ \rho \in \mathit{Topology}$

$\wedge\ \sigma \in \mathit{Topology}$

$\wedge\ \tau \in \mathit{Topology}$

$\wedge\ \upsilon \in \mathit{Topology}$

$\Rightarrow (\lambda\ (x,\ y)\bullet\ (f\ x,\ g\ y))$

$\in (\rho \times_T \tau,\ \sigma \times_T \upsilon)\ \mathit{Homeomorphism}$

product_unit_homeomorphism_thm

$\vdash \forall\ \tau$

$\bullet\ \tau \in \mathit{Topology}$

$\Rightarrow (\lambda\ x\bullet\ (x,\ \mathit{One})) \in (\tau,\ \tau \times_T 1_T)\ \mathit{Homeomorphism}$

swap_homeomorphism_thm

$\vdash \forall\ \sigma\ \tau$

$\bullet\ \sigma \in \mathit{Topology} \wedge \tau \in \mathit{Topology}$

$\Rightarrow (\lambda\ (x,\ y)\bullet\ (y,\ x))$

$\in (\sigma \times_T \tau,\ \tau \times_T \sigma)\ \mathit{Homeomorphism}$

homeomorphism_open_mapping_thm

$\vdash \forall\ f\ \sigma\ \tau\ A$

$\bullet\ f \in (\sigma,\ \tau)\ \mathit{Homeomorphism}$

$\wedge\ A \in \sigma$

$\wedge\ \sigma \in \mathit{Topology}$

$\wedge\ \tau \in \mathit{Topology}$

$\Rightarrow \{y|\exists\ x\bullet\ x \in A \wedge y = f\ x\} \in \tau$

homeomorphism_closed_mapping_thm

$\vdash \forall\ f\ \sigma\ \tau\ A$

$\bullet\ f \in (\sigma,\ \tau)\ \mathit{Homeomorphism}$

$\wedge\ A \in \sigma\ \mathit{Closed}$

$\wedge\ \sigma \in \mathit{Topology}$

$\wedge\ \tau \in \mathit{Topology}$

$\Rightarrow \{y|\exists\ x\bullet\ x \in A \wedge y = f\ x\} \in \tau\ \mathit{Closed}$

homeomorphism_one_one_thm

$\vdash \forall\ f\ \sigma\ \tau\ x\ y$

$\bullet\ f \in (\sigma,\ \tau)\ \mathit{Homeomorphism}$

$\wedge\ \sigma \in \mathit{Topology}$

$\wedge\ \tau \in \mathit{Topology}$

$\wedge\ x \in \mathit{Space}_T\ \sigma$

$\wedge\ y \in \mathit{Space}_T\ \sigma$

$\wedge\ f\ x = f\ y$

$\Rightarrow x = y$

homeomorphism_onto_thm

$\vdash \forall\ f\ \sigma\ \tau\ y$

$\bullet\ f \in (\sigma,\ \tau)\ \mathit{Homeomorphism}$

$\wedge\ \sigma \in \mathit{Topology}$

$\wedge\ \tau \in \mathit{Topology}$

$\wedge\ y \in \mathit{Space}_T\ \tau$

$\Rightarrow (\exists\ x\bullet\ x \in \mathit{Space}_T\ \sigma \wedge y = f\ x)$

homeomorphism_one_one_open_mapping_thm

$\vdash \forall f\ \sigma\ \tau$
$\bullet\ \sigma \in Topology \land \tau \in Topology$
$\Rightarrow (f \in (\sigma, \tau)\ Homeomorphism$
$\Leftrightarrow (\forall\ x\ y$
$\bullet\ x \in Space_T\ \sigma \land y \in Space_T\ \sigma \land f\ x = f\ y$
$\Rightarrow x = y)$
$\land (\forall\ y$
$\bullet\ y \in Space_T\ \tau$
$\Rightarrow (\exists\ x \bullet\ x \in Space_T\ \sigma \land y = f\ x))$
$\land f \in (\sigma, \tau)\ Continuous$
$\land (\forall\ A$
$\bullet\ A \in \sigma \Rightarrow \{y | \exists\ x \bullet\ x \in A \land y = f\ x\} \in \tau))$

homeomorphism_one_one_closed_mapping_thm

$\vdash \forall f\ \sigma\ \tau$
$\bullet\ \sigma \in Topology \land \tau \in Topology$
$\Rightarrow (f \in (\sigma, \tau)\ Homeomorphism$
$\Leftrightarrow (\forall\ x\ y$
$\bullet\ x \in Space_T\ \sigma \land y \in Space_T\ \sigma \land f\ x = f\ y$
$\Rightarrow x = y)$
$\land (\forall\ y$
$\bullet\ y \in Space_T\ \tau$
$\Rightarrow (\exists\ x \bullet\ x \in Space_T\ \sigma \land y = f\ x))$
$\land f \in (\sigma, \tau)\ Continuous$
$\land (\forall\ A$
$\bullet\ A \in \sigma\ Closed$
$\Rightarrow \{y | \exists\ x \bullet\ x \in A \land y = f\ x\} \in \tau\ Closed))$

⊆_compact_homeomorphism_thm

$\vdash \forall f\ \sigma\ \tau\ B\ C$
$\bullet\ \sigma \in Topology$
$\land \sigma \in Hausdorff$
$\land \tau \in Topology$
$\land \tau \in Hausdorff$
$\land C \in \sigma\ Compact$
$\land B \subseteq C$
$\land f \in (\sigma, \tau)\ Continuous$
$\land (\forall\ x\ y \bullet\ x \in B \land y \in C \land f\ x = f\ y \Rightarrow x = y)$
$\Rightarrow f$
$\in (B \lhd_T \sigma,$
$\{y | \exists\ x \bullet\ x \in B \land y = f\ x\}$
$\lhd_T \tau)\ Homeomorphism$

interior_boundary_⊆_space_t_thm

$\vdash \forall \tau\ A$
$\bullet\ \tau\ Interior\ A \subseteq Space_T\ \tau \land \tau\ Boundary\ A \subseteq Space_T\ \tau$

interior_⊆_thm

$\vdash \forall \tau\ A \bullet\ \tau\ Interior\ A \subseteq A$

boundary_interior_thm

$\vdash \forall \tau\ A$
$\bullet\ \tau \in Topology$
$\Rightarrow \tau\ Boundary\ A$
$= Space_T\ \tau$

$\setminus$ (τ *Interior* $A \cup \tau$ *Interior* ($Space_T$ $\tau \setminus A$))

interior_×_thm

$\vdash \forall\ \sigma\ \tau\ A\ B$
• $(\sigma \times_T \tau)$ *Interior* $(A \times B)$
= (σ *Interior* $A \times \tau$ *Interior* B)

open_⇔_disjoint_boundary_thm

$\vdash \forall\ \tau\ A$
• $\tau \in$ *Topology*
⇒ ($A \in \tau \Leftrightarrow A \subseteq Space_T\ \tau \wedge A \cap \tau$ *Boundary* $A = \{\}$)

closed_⇔_boundary_⊆_thm

$\vdash \forall\ \tau\ A$
• $\tau \in$ *Topology*
⇒ ($A \in \tau$ *Closed*
⇔ $A \subseteq Space_T\ \tau \wedge \tau$ *Boundary* $A \subseteq A$)

interior_⋃_thm

$\vdash \forall\ \tau\ A$
• $\tau \in$ *Topology* ⇒ τ *Interior* $A = \bigcup \{B | B \in \tau \wedge B \subseteq A\}$

closure_interior_complement_thm

$\vdash \forall\ \tau\ A$
• $\tau \in$ *Topology*
⇒ τ *Closure* A
= $Space_T\ \tau \setminus \tau$ *Interior* ($Space_T\ \tau \setminus A$)

unique_lifting_thm

$\vdash \forall\ \rho\ \sigma\ \tau\ p\ f\ g\ a$
• $\rho \in$ *Topology*
∧ $\sigma \in$ *Topology*
∧ $\tau \in$ *Topology*
∧ $Space_T\ \rho \in \rho$ *Connected*
∧ $p \in (\sigma, \tau)$ *CoveringProjection*
∧ $f \in (\rho, \sigma)$ *Continuous*
∧ $g \in (\rho, \sigma)$ *Continuous*
∧ ($\forall\ x \bullet\ x \in Space_T\ \rho \Rightarrow p\ (f\ x) = p\ (g\ x)$)
∧ $a \in Space_T\ \rho$
∧ $g\ a = f\ a$
⇒ ($\forall\ x \bullet\ x \in Space_T\ \rho \Rightarrow g\ x = f\ x$)

B THE THEORY metric_spaces

B.1 Parents

trees *analysis* *topology*

B.2 Children

topology_ℝ

B.3 Constants

Metric $('a \times 'a \to \mathbb{R})\ \mathbb{P}$
$**MetricTopology**
$('a \times 'a \to \mathbb{R}) \to 'a\ \mathbb{P}\ \mathbb{P}$
ListMetric $('a \times 'a \to \mathbb{R}) \to 'a\ LIST \times 'a\ LIST \to \mathbb{R}$

B.4 Fixity

Postfix 400: **MetricTopology**

B.5 Definitions

Metric $\vdash Metric$
$= \{D$
$|(\forall\ x\ y \bullet\ 0. \le D\ (x,\ y))$
$\land\ (\forall\ x\ y \bullet\ D\ (x,\ y) = 0. \Leftrightarrow x = y)$
$\land\ (\forall\ x\ y \bullet\ D\ (x,\ y) = D\ (y,\ x))$
$\land\ (\forall\ x\ y\ z \bullet\ D\ (x,\ z) \le D\ (x,\ y) + D\ (y,\ z))\}$

MetricTopology
$\vdash \forall\ D$
$\bullet\ D\ MetricTopology$
$= \{A$
$|\forall\ x$
$\bullet\ x \in A$
$\Rightarrow (\exists\ e$
$\bullet\ 0. < e \land (\forall\ y \bullet\ D\ (x,\ y) < e \Rightarrow y \in A))\}$

ListMetric $\vdash \forall\ D\ x\ v\ y\ w$
$\bullet\ ListMetric\ D\ ([],\ []) = 0.$
$\land\ ListMetric\ D\ (Cons\ x\ v,\ [])$
$= 1. + D\ (x,\ Arbitrary) + ListMetric\ D\ (v,\ [])$
$\land\ ListMetric\ D\ ([],\ Cons\ y\ w)$
$= 1. + D\ (Arbitrary,\ y) + ListMetric\ D\ ([],\ w)$
$\land\ ListMetric\ D\ (Cons\ x\ v,\ Cons\ y\ w)$
$= D\ (x,\ y) + ListMetric\ D\ (v,\ w)$

B.6 Theorems

metric_topology_thm

$\vdash \forall\ D \bullet\ D \in Metric \Rightarrow D\ MetricTopology \in Topology$

space_t_metric_topology_thm

$\vdash \forall\ D$
$\quad \bullet\ D \in Metric \Rightarrow Space_T\ (D\ MetricTopology) = Universe$

open_ball_open_thm

$\vdash \forall\ D\ e\ x$
$\quad \bullet\ 0. < e \wedge D \in Metric$
$\qquad \Rightarrow \{y | D\ (x,\ y) < e\} \in D\ MetricTopology$

open_ball_neighbourhood_thm

$\vdash \forall\ D\ e\ x \bullet\ 0. < e \wedge D \in Metric \Rightarrow x \in \{y | D\ (x,\ y) < e\}$

metric_topology_hausdorff_thm

$\vdash \forall\ D \bullet\ D \in Metric \Rightarrow D\ MetricTopology \in Hausdorff$

product_metric_thm

$\vdash \forall\ D1\ D2$
$\quad \bullet\ D1 \in Metric \wedge D2 \in Metric$
$\qquad \Rightarrow (\lambda\ ((x1,\ x2),\ y1,\ y2)$
$\qquad\quad \bullet\ D1\ (x1,\ y1) + D2\ (x2,\ y2))$
$\qquad\quad \in Metric$

product_metric_topology_thm

$\vdash \forall\ D1\ D2$
$\quad \bullet\ D1 \in Metric \wedge D2 \in Metric$
$\qquad \Rightarrow (\lambda\ ((x1,\ x2),\ y1,\ y2)$
$\qquad\qquad \bullet\ D1\ (x1,\ y1) + D2\ (x2,\ y2))\ MetricTopology$
$\qquad\quad = D1\ MetricTopology \times_T D2\ MetricTopology$

lebesgue_number_thm

$\vdash \forall\ D\ X\ U$
$\quad \bullet\ D \in Metric$
$\qquad\quad \wedge X \in D\ MetricTopology\ Compact$
$\qquad\quad \wedge U \subseteq D\ MetricTopology$
$\qquad\quad \wedge X \subseteq \bigcup U$
$\qquad \Rightarrow (\exists\ e$
$\qquad \bullet\ 0. < e$
$\qquad\quad \wedge (\forall\ x$
$\qquad\quad \bullet\ x \in X$
$\qquad\qquad \Rightarrow (\exists\ A$
$\qquad\qquad \bullet\ x \in A$
$\qquad\qquad\quad \wedge A \in U$
$\qquad\qquad\quad \wedge (\forall\ y \bullet\ D\ (x,\ y) < e \Rightarrow y \in A))))$

collar_thm

$\vdash \forall\ D\ X\ U$
$\quad \bullet\ D \in Metric$
$\qquad\quad \wedge X \in D\ MetricTopology\ Compact$
$\qquad\quad \wedge A \in D\ MetricTopology$
$\qquad\quad \wedge X \subseteq A$
$\qquad \Rightarrow (\exists\ e$
$\qquad \bullet\ 0. < e$
$\qquad\quad \wedge (\forall\ x\ y$
$\qquad\quad \bullet\ x \in X \wedge y \in Space_T\ \tau \wedge D\ (x,\ y) < e$
$\qquad\qquad \Rightarrow y \in A))$

list_metric_nonneg_thm

$\vdash \forall\ D\ x \bullet\ D \in Metric \Rightarrow 0. \leq ListMetric\ D\ (x,\ y)$

list_metric_sym_thm

$\vdash \forall\ D\ x\ y$

$\bullet\ D \in Metric$

$\Rightarrow ListMetric\ D\ (x,\ y) = ListMetric\ D\ (y,\ x)$

list_metric_metric_thm

$\vdash \forall\ D \bullet\ D \in Metric \Rightarrow ListMetric\ D \in Metric$

C THE THEORY topology_$\mathbb{R}$

C.1 Parents

metric_spaces

C.2 Children

homotopy

C.3 Constants

$\boldsymbol{D_R}$	$\mathbb{R} \times \mathbb{R} \to \mathbb{R}$
$\boldsymbol{D_{R2}}$	$(\mathbb{R} \times \mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
\$***Space***	$\mathbb{N} \to \mathbb{R}$ *LIST* $\mathbb{P}$ $\mathbb{P}$
\$***Cube***	$\mathbb{N} \to \mathbb{R}$ *LIST* $\mathbb{P}$ $\mathbb{P}$
\$***OpenCube***	$\mathbb{N} \to \mathbb{R}$ *LIST* $\mathbb{P}$ $\mathbb{P}$
\$***Sphere***	$\mathbb{N} \to \mathbb{R}$ *LIST* $\mathbb{P}$ $\mathbb{P}$

C.4 Aliases

$\boldsymbol{O_R}$	$Open_R : \mathbb{R}\ \mathbb{P}\ \mathbb{P}$

C.5 Fixity

Postfix 400: ***Cube*** ***OpenCube*** ***Space*** ***Sphere***

C.6 Definitions

$\boldsymbol{D_R}$ $\vdash \forall\ x\ y \bullet D_R\ (x,\ y) = Abs\ (y - x)$

$\boldsymbol{D_{R2}}$ $\vdash \forall\ x1\ y1\ x2\ y2$
$\bullet\ D_{R2}\ ((x1,\ y1),\ x2,\ y2)$
$= Abs\ (x2 - x1) + Abs\ (y2 - y1)$

Space $\vdash \forall\ n$
$\bullet\ n\ Space$
$= \{v | \#\ v = n\} \lhd_T ListMetric\ D_R\ MetricTopology$

Cube $\vdash \forall\ n$
$\bullet\ n\ Cube$
$= \{v | Elems\ v \subseteq ClosedInterval\ 0.\ 1.\} \lhd_T n\ Space$

OpenCube $\vdash \forall\ n$
$\bullet\ n\ OpenCube$
$= \{v | Elems\ v \subseteq OpenInterval\ 0.\ 1.\} \lhd_T n\ Space$

Sphere $\vdash \forall\ n$
$\bullet\ n\ Sphere = \{v | \neg\ Elems\ v \cap \{0.;\ 1.\} = \{\}\} \lhd_T n\ Cube$

C.7 Theorems

d_ℝ_2_def1
$$\vdash \forall\ xy1\ xy2 \bullet D_{R2}\ (xy1,\ xy2) = Abs\ (Fst\ xy2 - Fst\ xy1) + Abs\ (Snd\ xy2 - Snd\ xy1)$$

open_ℝ_topology_thm
$$\vdash O_R \in Topology$$

space_t_ℝ_thm
$$\vdash Space_T\ O_R = Universe$$

closed_closed_ℝ_thm
$$\vdash O_R\ Closed = Closed_R$$

compact_compact_ℝ_thm
$$\vdash O_R\ Compact = Compact_R$$

open_ℝ_const_continuous_thm
$$\vdash \forall\ \sigma\ c \bullet\ \sigma \in Topology \Rightarrow (\lambda\ x \bullet\ c) \in (\sigma,\ O_R)\ Continuous$$

open_ℝ_id_continuous_thm
$$\vdash (\lambda\ x \bullet\ x) \in (O_R,\ O_R)\ Continuous$$

continuous_cts_at_ℝ_thm
$$\vdash \forall\ f \bullet\ f \in (O_R,\ O_R)\ Continuous \Leftrightarrow (\forall\ x \bullet\ f\ Cts\ x)$$

cts_at_ℝ_continuous_thm
$$\vdash \forall\ f \bullet\ (\forall\ x \bullet\ f\ Cts\ x) \Leftrightarrow f \in (O_R,\ O_R)\ Continuous$$

universe_ℝ_connected_thm
$$\vdash Universe \in O_R\ Connected$$

closed_interval_connected_thm
$$\vdash \forall\ x\ y \bullet\ x < y \Rightarrow ClosedInterval\ x\ y \in O_R\ Connected$$

connected_ℝ_thm
$$\vdash \forall\ X \bullet\ X \in O_R\ Connected \Leftrightarrow (\forall\ x\ y\ z \bullet\ x \in X \land y \in X \land x \le z \land z \le y \Rightarrow z \in X)$$

continuous_ℝ_×_ℝ_ℝ_thm
$$\vdash \forall\ X\ f \bullet\ X \in O_R \times_T O_R \Rightarrow (f \in (X \lhd_T O_R \times_T O_R,\ O_R)\ Continuous \Leftrightarrow (\forall\ x\ y\ u\ v \bullet\ f\ (u,\ v) \in OpenInterval\ x\ y \land (u,\ v) \in X \Rightarrow (\exists\ a\ b\ c\ d \bullet\ u \in OpenInterval\ a\ b \land v \in OpenInterval\ c\ d \land (\forall\ s\ t \bullet\ s \in OpenInterval\ a\ b \land t \in OpenInterval\ c\ d \land (s,\ t) \in X \Rightarrow f\ (s,\ t) \in OpenInterval\ x\ y))))$$

continuous_ℝ_×_ℝ_ℝ_thm1
$$\vdash \forall\ f \bullet\ f \in (O_R \times_T O_R,\ O_R)\ Continuous \Leftrightarrow (\forall\ x\ y\ u\ v \bullet\ f\ (u,\ v) \in OpenInterval\ x\ y \Rightarrow (\exists\ a\ b\ c\ d \bullet\ u \in OpenInterval\ a\ b$$

$$\land\ v \in OpenInterval\ c\ d$$
$$\land\ (\forall\ s\ t$$
$$\bullet\ s \in OpenInterval\ a\ b$$
$$\land\ t \in OpenInterval\ c\ d$$
$$\Rightarrow f\ (s,\ t) \in OpenInterval\ x\ y)))$$

continuous_ℝ_×_ℝ_ℝ_thm3

$$\vdash \forall\ X\ f$$
$$\bullet\ X \in O_R \times_T O_R$$
$$\Rightarrow (f \in (X \lhd_T O_R \times_T O_R,\ O_R)\ Continuous$$
$$\Leftrightarrow (\forall\ e\ u\ v$$
$$\bullet\ 0.\ <\ e \land (u,\ v) \in X$$
$$\Rightarrow (\exists\ d1\ d2$$
$$\bullet\ 0.\ <\ d1$$
$$\land\ 0.\ <\ d2$$
$$\land\ (\forall\ s\ t$$
$$\bullet\ Abs\ (s + \sim u) < d1$$
$$\land\ Abs\ (t + \sim v) < d2$$
$$\land\ (s,\ t) \in X$$
$$\Rightarrow Abs\ (f\ (s,\ t) + \sim (f\ (u,\ v)))$$
$$<\ e))))$$

continuous_ℝ_×_ℝ_ℝ_thm4

$$\vdash \forall\ f$$
$$\bullet\ f \in (O_R \times_T O_R,\ O_R)\ Continuous$$
$$\Leftrightarrow (\forall\ e\ u\ v$$
$$\bullet\ 0.\ <\ e$$
$$\Rightarrow (\exists\ d1\ d2$$
$$\bullet\ 0.\ <\ d1$$
$$\land\ 0.\ <\ d2$$
$$\land\ (\forall\ s\ t$$
$$\bullet\ Abs\ (s + \sim u) < d1 \land Abs\ (t + \sim v) < d2$$
$$\Rightarrow Abs\ (f\ (s,\ t) + \sim (f\ (u,\ v)))$$
$$<\ e)))$$

plus_continuous_ℝ_×_ℝ_thm

$$\vdash Uncurry\ \$+ \in (O_R \times_T O_R,\ O_R)\ Continuous$$

times_continuous_ℝ_×_ℝ_thm

$$\vdash Uncurry\ \$* \in (O_R \times_T O_R,\ O_R)\ Continuous$$

cond_continuous_ℝ_thm

$$\vdash \forall\ b\ c\ f\ g\ \sigma\ \tau$$
$$\bullet\ \sigma \in Topology$$
$$\land\ \tau \in Topology$$
$$\land\ c \in (\sigma,\ O_R)\ Continuous$$
$$\land\ f \in (\sigma,\ \tau)\ Continuous$$
$$\land\ g \in (\sigma,\ \tau)\ Continuous$$
$$\land\ (\forall\ x \bullet\ x \in Space_T\ \sigma \land c\ x = b \Rightarrow f\ x = g\ x)$$
$$\Rightarrow (\lambda\ x \bullet\ if\ c\ x \leq b\ then\ f\ x\ else\ g\ x)$$
$$\in (\sigma,\ \tau)\ Continuous$$

d_ℝ_metric_thm

$$\vdash D_R \in Metric$$

d_ℝ_open_ℝ_thm

$$\vdash D_R\ MetricTopology = O_R$$

d_ℝ_2_metric_thm

$$\vdash D_{R2} \in Metric$$

d_ℝ_2_open_ℝ_×_open_ℝ_thm

$$\vdash D_{R2}\ MetricTopology = O_R \times_T O_R$$

open_ℝ_hausdorff_thm

$$\vdash O_R \in Hausdorff$$

open_ℝ_×_open_ℝ_hausdorff_thm

$$\vdash O_R \times_T O_R \in Hausdorff$$

ℝ_lebesgue_number_thm

$$\begin{array}{l}
\vdash \forall\ X\ U \\
\quad \bullet\ X \in Compact_R \land U \subseteq O_R \land X \subseteq \bigcup U \\
\qquad \Rightarrow (\exists\ e \\
\qquad \bullet\ 0. < e \\
\qquad\quad \land (\forall\ x \\
\qquad\quad \bullet\ x \in X \\
\qquad\qquad \Rightarrow (\exists\ A \\
\qquad\qquad \bullet\ x \in A \\
\qquad\qquad\quad \land A \in U \\
\qquad\qquad\quad \land (\forall\ y \bullet\ Abs\ (y - x) < e \Rightarrow y \in A))))
\end{array}$$

closed_interval_lebesgue_number_thm

$$\begin{array}{l}
\vdash \forall\ y\ z\ U \\
\quad \bullet\ U \subseteq O_R \land ClosedInterval\ y\ z \subseteq \bigcup U \\
\qquad \Rightarrow (\exists\ e \\
\qquad \bullet\ 0. < e \\
\qquad\quad \land (\forall\ x \\
\qquad\quad \bullet\ x \in ClosedInterval\ y\ z \\
\qquad\qquad \Rightarrow (\exists\ A \\
\qquad\qquad \bullet\ x \in A \\
\qquad\qquad\quad \land A \in U \\
\qquad\qquad\quad \land (\forall\ y \bullet\ Abs\ (y - x) < e \Rightarrow y \in A))))
\end{array}$$

dissect_unit_interval_thm

$$\begin{array}{l}
\vdash \forall\ x \\
\quad \bullet\ 0. < x \\
\qquad \Rightarrow (\exists\ n\ t \\
\qquad \bullet\ 0 < n \\
\qquad\quad \land t\ 0 = 0. \\
\qquad\quad \land t\ n = 1. \\
\qquad\quad \land (\forall\ i\ j \bullet\ i < j \Rightarrow t\ i < t\ j) \\
\qquad\quad \land (\forall\ i \bullet\ t\ (i + 1) - t\ i < x))
\end{array}$$

***product_interval_cover_thm*1**

$$\begin{array}{l}
\vdash \forall\ \tau\ U\ x \\
\quad \bullet\ \tau \in Topology \\
\qquad \land U \subseteq \tau \times_T O_R \\
\qquad \land x \in Space_T\ \tau \\
\qquad \land (\forall\ s \\
\qquad \bullet\ s \in ClosedInterval\ 0.\ 1. \\
\qquad\quad \Rightarrow (\exists\ B \bullet\ (x, s) \in B \land B \in U)) \\
\qquad \Rightarrow (\exists\ n\ t\ A \\
\qquad \bullet\ t\ 0 = 0. \\
\qquad\quad \land t\ n = 1. \\
\qquad\quad \land (\forall\ i \bullet\ t\ i < t\ (i + 1)) \\
\qquad\quad \land x \in A
\end{array}$$

$\land\ A \in \tau$
$\land\ (\forall\ i$
$\bullet\ i < n$
$\Rightarrow (\exists\ B$
$\bullet\ B \in U$
$\land\ (A$
$\times\ ClosedInterval$
$(t\ i)$
$(t\ (i + 1)))$
$\subseteq B)))$

inc_seq_thm $\vdash \forall\ t\ i\ j$
$\bullet\ (\forall\ i \bullet\ t\ i < t\ (i + 1)) \Leftrightarrow (\forall\ i\ j \bullet\ i < j \Rightarrow t\ i < t\ j)$

product_interval_cover_thm
$\vdash \forall\ \tau\ U\ x$
$\bullet\ \tau \in Topology$
$\land\ U \subseteq \tau \times_T O_R$
$\land\ x \in Space_T\ \tau$
$\land\ (\forall\ s$
$\bullet\ s \in ClosedInterval\ 0.\ 1.$
$\Rightarrow (\exists\ B \bullet\ (x,\ s) \in B \land B \in U))$
$\Rightarrow (\exists\ n\ t\ A$
$\bullet\ t\ 0 = 0.$
$\land\ t\ n = 1.$
$\land\ (\forall\ i\ j \bullet\ i < j \Rightarrow t\ i < t\ j)$
$\land\ x \in A$
$\land\ A \in \tau$
$\land\ (\forall\ i$
$\bullet\ i < n$
$\Rightarrow (\exists\ B$
$\bullet\ B \in U$
$\land\ (A$
$\times\ ClosedInterval$
$(t\ i)$
$(t\ (i + 1)))$
$\subseteq B)))$

D THE THEORY homotopy

D.1 Parents

$topology\_\mathbb{R}$

D.2 Constants

Paths $'a\ \mathbb{P}\ \mathbb{P} \to (\mathbb{R} \to 'a)\ \mathbb{P}$

$\$$**PathConnected**
$'a\ \mathbb{P}\ \mathbb{P} \to 'a\ \mathbb{P}\ \mathbb{P}$

LocallyPathConnected
$'a\ \mathbb{P}\ \mathbb{P}\ \mathbb{P}$

$\$$**Homotopy** $'a\ \mathbb{P}\ \mathbb{P} \times 'a\ \mathbb{P} \times 'b\ \mathbb{P}\ \mathbb{P} \to ('a \times \mathbb{R} \to 'b)\ \mathbb{P}$

$\$$**HomotopyClass**
$'a\ \mathbb{P}\ \mathbb{P} \times 'a\ \mathbb{P} \times 'b\ \mathbb{P}\ \mathbb{P} \to ('a \to 'b) \to ('a \to 'b)\ \mathbb{P}$

$\$+_P$ $(\mathbb{R} \to 'a) \to (\mathbb{R} \to 'a) \to \mathbb{R} \to 'a$

$\mathbf{0}_P$ $'a \to \mathbb{R} \to 'a$

$\sim_P$ $(\mathbb{R} \to 'a) \to \mathbb{R} \to 'a$

HomotopyLiftingProperty
$'a\ \mathbb{P}\ \mathbb{P} \leftrightarrow (('b \to 'c) \times 'b\ \mathbb{P}\ \mathbb{P} \times 'c\ \mathbb{P}\ \mathbb{P})$

D.3 Fixity

Right Infix 300:
$+_P$

Postfix 400: **Homotopy** **HomotopyClass** **PathConnected**

D.4 Definitions

Paths
$\vdash \forall\ \tau$
$\bullet\ Paths\ \tau$
$= \{f$
$|f \in (O_R, \tau)\ Continuous$
$\wedge\ (\forall\ x \bullet\ x \le 0. \Rightarrow f\ x = f\ 0.)$
$\wedge\ (\forall\ x \bullet\ 1. \le x \Rightarrow f\ x = f\ 1.)\}$

PathConnected
$\vdash \forall\ \tau$
$\bullet\ \tau\ PathConnected$
$= \{A$
$|A \subseteq Space_T\ \tau$
$\wedge\ (\forall\ x\ y$
$\bullet\ x \in A \wedge y \in A$
$\Rightarrow (\exists\ f$
$\bullet\ f \in Paths\ \tau$
$\wedge\ (\forall\ t \bullet\ f\ t \in A)$
$\wedge\ f\ 0. = x$
$\wedge\ f\ 1. = y))\}$

LocallyPathConnected
$\vdash \forall\ \tau$

$\bullet\ \tau \in \mathit{LocallyPathConnected}$
$\Leftrightarrow (\forall\ x\ A$
$\bullet\ x \in A \wedge A \in \tau$
$\Rightarrow (\exists\ B$
$\bullet\ B \in \tau$
$\wedge\ x \in B$
$\wedge\ B \subseteq A$
$\wedge\ B \in \tau\ \mathit{PathConnected}))$

Homotopy $\vdash \forall\ \sigma\ X\ \tau$
$\bullet\ (\sigma,\ X,\ \tau)\ \mathit{Homotopy}$
$= \{f$
$| f \in (\sigma \times_T O_R,\ \tau)\ \mathit{Continuous}$
$\wedge\ (\forall\ x\ s\ t \bullet\ x \in X \Rightarrow f\ (x,\ s) = f\ (x,\ t))\}$

HomotopyClass
$\vdash \forall\ \sigma\ X\ \tau\ f$
$\bullet\ ((\sigma,\ X,\ \tau)\ \mathit{HomotopyClass})\ f$
$= \{g$
$| \exists\ H$
$\bullet\ H \in (\sigma,\ X,\ \tau)\ \mathit{Homotopy}$
$\wedge\ (\forall\ x \bullet\ H\ (x,\ 0.) = f\ x)$
$\wedge\ (\forall\ x \bullet\ H\ (x,\ 1.) = g\ x)\}$

$\boldsymbol{+_P}$ $\vdash \forall\ f\ g$
$\bullet\ f +_P g$
$= (\lambda\ t$
$\bullet\ \mathit{if}\ t \leq 1\ /\ 2$
$\mathit{then}\ f\ (2. * t)$
$\mathit{else}\ g\ (2. * (t - 1\ /\ 2)))$

$\boldsymbol{0_P}$ $\vdash \forall\ x \bullet\ 0_P\ x = (\lambda\ t \bullet\ x)$

$\boldsymbol{\sim_P}$ $\vdash \forall\ f \bullet\ \sim_P f = (\lambda\ t \bullet\ f\ (1. - t))$

HomotopyLiftingProperty
$\vdash \forall\ \rho\ \sigma\ \tau\ p$
$\bullet\ (\rho,\ p,\ \sigma,\ \tau) \in \mathit{HomotopyLiftingProperty}$
$\Leftrightarrow \rho \in \mathit{Topology}$
$\wedge\ \sigma \in \mathit{Topology}$
$\wedge\ \tau \in \mathit{Topology}$
$\wedge\ p \in (\sigma,\ \tau)\ \mathit{Continuous}$
$\wedge\ (\forall\ f\ h$
$\bullet\ f \in (\rho,\ \sigma)\ \mathit{Continuous}$
$\wedge\ h \in (\rho \times_T O_R,\ \tau)\ \mathit{Continuous}$
$\wedge\ (\forall\ x$
$\bullet\ x \in \mathit{Space}_T\ \rho \Rightarrow h\ (x,\ 0.) = p\ (f\ x))$
$\Rightarrow (\exists\ L$
$\bullet\ L \in (\rho \times_T O_R,\ \sigma)\ \mathit{Continuous}$
$\wedge\ (\forall\ x$
$\bullet\ x \in \mathit{Space}_T\ \rho \Rightarrow L\ (x,\ 0.) = f\ x)$
$\wedge\ (\forall\ x\ s$
$\bullet\ x \in \mathit{Space}_T\ \rho$
$\wedge\ s \in \mathit{ClosedInterval}\ 0.\ 1.$
$\Rightarrow p\ (L\ (x,\ s)) = h\ (x,\ s))))$

D.5 Theorems

path_connected_connected_thm

$\vdash \forall\ \tau\ X$
$\quad \bullet\ \tau \in \mathit{Topology} \land X \in \tau\ \mathit{PathConnected}$
$\quad\quad \Rightarrow X \in \tau\ \mathit{Connected}$

product_path_connected_thm

$\vdash \forall\ \sigma\ \tau\ X\ Y$
$\quad \bullet\ \sigma \in \mathit{Topology}$
$\quad\quad \land\ \tau \in \mathit{Topology}$
$\quad\quad \land\ X \in \sigma\ \mathit{PathConnected}$
$\quad\quad \land\ Y \in \tau\ \mathit{PathConnected}$
$\quad \Rightarrow (X \times Y) \in (\sigma \times_T \tau)\ \mathit{PathConnected}$

homotopy_class_refl_thm

$\vdash \forall\ \sigma\ X\ \tau\ f$
$\quad \bullet\ \sigma \in \mathit{Topology} \land \tau \in \mathit{Topology} \land f \in (\sigma,\ \tau)\ \mathit{Continuous}$
$\quad\quad \Rightarrow f \in ((\sigma,\ X,\ \tau)\ \mathit{HomotopyClass})\ f$

homotopy_class_sym_thm

$\vdash \forall\ \sigma\ X\ \tau\ f\ g$
$\quad \bullet\ \sigma \in \mathit{Topology}$
$\quad\quad \land\ \tau \in \mathit{Topology}$
$\quad\quad \land\ g \in ((\sigma,\ X,\ \tau)\ \mathit{HomotopyClass})\ f$
$\quad \Rightarrow f \in ((\sigma,\ X,\ \tau)\ \mathit{HomotopyClass})\ g$

homotopy_class_trans_thm

$\vdash \forall\ \sigma\ X\ \tau\ f\ g\ h$
$\quad \bullet\ \sigma \in \mathit{Topology}$
$\quad\quad \land\ \tau \in \mathit{Topology}$
$\quad\quad \land\ g \in ((\sigma,\ X,\ \tau)\ \mathit{HomotopyClass})\ f$
$\quad\quad \land\ h \in ((\sigma,\ X,\ \tau)\ \mathit{HomotopyClass})\ g$
$\quad \Rightarrow h \in ((\sigma,\ X,\ \tau)\ \mathit{HomotopyClass})\ f$

homotopy_⊆_thm

$\vdash \forall\ \sigma\ X\ Y\ \tau\ H$
$\quad \bullet\ \sigma \in \mathit{Topology}$
$\quad\quad \land\ \tau \in \mathit{Topology}$
$\quad\quad \land\ H \in (\sigma,\ X,\ \tau)\ \mathit{Homotopy}$
$\quad\quad \land\ Y \subseteq X$
$\quad \Rightarrow H \in (\sigma,\ Y,\ \tau)\ \mathit{Homotopy}$

homotopy_class_⊆_thm

$\vdash \forall\ \sigma\ X\ Y\ \tau\ f\ g$
$\quad \bullet\ \sigma \in \mathit{Topology}$
$\quad\quad \land\ \tau \in \mathit{Topology}$
$\quad\quad \land\ g \in ((\sigma,\ X,\ \tau)\ \mathit{HomotopyClass})\ f$
$\quad\quad \land\ Y \subseteq X$
$\quad \Rightarrow g \in ((\sigma,\ Y,\ \tau)\ \mathit{HomotopyClass})\ f$

homotopy_class_comp_left_thm

$\vdash \forall\ \rho\ \sigma\ \tau\ X\ f\ g\ h$
$\quad \bullet\ \rho \in \mathit{Topology}$
$\quad\quad \land\ \sigma \in \mathit{Topology}$
$\quad\quad \land\ \tau \in \mathit{Topology}$
$\quad\quad \land\ g \in ((\rho,\ X,\ \sigma)\ \mathit{HomotopyClass})\ f$
$\quad\quad \land\ h \in (\sigma,\ \tau)\ \mathit{Continuous}$
$\quad \Rightarrow (\lambda\ x \bullet\ h\ (g\ x))$

$\in ((\rho,\ X,\ \tau)\ HomotopyClass)\ (\lambda\ x\bullet\ h\ (f\ x))$

homotopy_class_comp_right_thm

$\vdash \forall\ \rho\ \sigma\ \tau\ X\ f\ g\ h$
$\bullet\ \rho \in\ Topology$
$\wedge\ \sigma \in\ Topology$
$\wedge\ \tau \in\ Topology$
$\wedge\ g \in ((\sigma,\ X,\ \tau)\ HomotopyClass)\ f$
$\wedge\ h \in (\rho,\ \sigma)\ Continuous$
$\Rightarrow (\lambda\ x\bullet\ g\ (h\ x))$
$\in ((\rho,\ \{x | h\ x \in X\},\ \tau)\ HomotopyClass)$
$(\lambda\ x\bullet\ f\ (h\ x))$

homotopy_class_ℝ_thm

$\vdash \forall\ \tau\ f\ g$
$\bullet\ \tau \in\ Topology$
$\wedge\ f \in (\tau,\ O_R)\ Continuous$
$\wedge\ g \in (\tau,\ O_R)\ Continuous$
$\Rightarrow g \in ((\tau,\ \{x | g\ x = f\ x\},\ O_R)\ HomotopyClass)\ f$

half_open_interval_retract_thm

$\vdash \forall\ b$
$\bullet\ (\lambda\ s\bullet\ if\ s \le b\ then\ s\ else\ b)$
$\in (O_R,\ \{s | s \le b\} \lhd_T O_R)\ Continuous$

closed_interval_retract_thm

$\vdash \forall\ a\ b$
$\bullet\ a \le b$
$\Rightarrow (\lambda\ s$
$\bullet\ if\ s \le a\ then\ a\ else\ if\ s \le b\ then\ s\ else\ b)$
$\in (O_R,\ ClosedInterval\ a\ b \lhd_T O_R)\ Continuous$

×_closed_interval_retract_thm

$\vdash \forall\ \tau\ X\ a\ b$
$\bullet\ \tau \in\ Topology \wedge X \subseteq Space_T\ \tau \wedge a \le b$
$\Rightarrow (\lambda\ (x,\ s)$
$\bullet\ (x,$
$(if\ s \le a$
$then\ a$
$else\ if\ s \le b$
$then\ s$
$else\ b)))$
$\in ((X \times Universe) \lhd_T \tau \times_T O_R,$
$(X \times ClosedInterval\ a\ b)$
$\lhd_T \tau \times_T O_R)\ Continuous$

closed_interval_extension_thm

$\vdash \forall\ \rho\ \sigma\ f\ X\ a\ b$
$\bullet\ \rho \in\ Topology$
$\wedge\ \sigma \in\ Topology$
$\wedge\ X \subseteq Space_T\ \rho$
$\wedge\ a \le b$
$\wedge\ f$
$\in ((X \times ClosedInterval\ a\ b) \lhd_T \rho \times_T O_R,$
$\sigma)\ Continuous$
$\Rightarrow (\exists\ g$
$\bullet\ g$

$\in ((X \times \mathit{Universe}) \lhd_T \rho \times_T O_R,$
$\sigma)\ \mathit{Continuous}$
$\land (\forall\ x\ s$
$\bullet\ x \in X \land s \in \mathit{ClosedInterval}\ a\ b$
$\Rightarrow g\ (x,\ s) = f\ (x,\ s)))$

×_interval_glueing_thm

$\vdash \forall\ \rho\ \sigma\ f\ g\ X\ a\ b$
$\bullet\ \rho \in \mathit{Topology}$
$\land\ \sigma \in \mathit{Topology}$
$\land\ X \subseteq \mathit{Space}_T\ \rho$
$\land\ a \leq b$
$\land\ b \leq c$
$\land\ f$
$\in ((X \times \mathit{ClosedInterval}\ a\ b) \lhd_T \rho \times_T O_R,$
$\sigma)\ \mathit{Continuous}$
$\land\ g$
$\in ((X \times \mathit{ClosedInterval}\ b\ c) \lhd_T \rho \times_T O_R,$
$\sigma)\ \mathit{Continuous}$
$\land (\forall\ x \bullet\ x \in X \Rightarrow f\ (x,\ b) = g\ (x,\ b))$
$\Rightarrow (\exists\ h$
$\bullet\ h$
$\in ((X \times \mathit{ClosedInterval}\ a\ c) \lhd_T \rho \times_T O_R,$
$\sigma)\ \mathit{Continuous}$
$\land (\forall\ x\ s$
$\bullet\ x \in X \land s \in \mathit{ClosedInterval}\ a\ b$
$\Rightarrow h\ (x,\ s) = f\ (x,\ s))$
$\land (\forall\ x\ s$
$\bullet\ x \in X \land s \in \mathit{ClosedInterval}\ b\ c$
$\Rightarrow h\ (x,\ s) = g\ (x,\ s)))$

paths_continuous_thm

$\vdash \forall\ \tau\ f$
$\bullet\ \tau \in \mathit{Topology} \land f \in \mathit{Paths}\ \tau$
$\Rightarrow f \in (O_R,\ \tau)\ \mathit{Continuous}$

paths_representative_thm

$\vdash \forall\ \tau\ f$
$\bullet\ \tau \in \mathit{Topology} \land f \in (O_R,\ \tau)\ \mathit{Continuous}$
$\Rightarrow (\exists_1\ g$
$\bullet\ g \in \mathit{Paths}\ \tau$
$\land (\forall\ s$
$\bullet\ s \in \mathit{ClosedInterval}\ 0.\ 1. \Rightarrow g\ s = f\ s))$

path_0_path_thm

$\vdash \forall\ \tau\ x \bullet\ \tau \in \mathit{Topology} \land x \in \mathit{Space}_T\ \tau \Rightarrow 0_P\ x \in \mathit{Paths}\ \tau$

path_plus_path_thm

$\vdash \forall\ \tau\ f\ g$
$\bullet\ \tau \in \mathit{Topology}$
$\land\ f \in \mathit{Paths}\ \tau$
$\land\ g \in \mathit{Paths}\ \tau$
$\land\ g\ 0. = f\ 1.$
$\Rightarrow f +_P g \in \mathit{Paths}\ \tau$

path_minus_path_thm

$\vdash \forall\ \tau\ f \bullet\ \tau \in \mathit{Topology} \land f \in \mathit{Paths}\ \tau \Rightarrow \sim_P f \in \mathit{Paths}\ \tau$

path_plus_assoc_thm

$\vdash \forall\ \tau\ f\ g\ h$
$\bullet\ \tau \in Topology$
$\wedge\ f \in Paths\ \tau$
$\wedge\ g \in Paths\ \tau$
$\wedge\ h \in Paths\ \tau$
$\wedge\ g\ 0. = f\ 1.$
$\wedge\ h\ 0. = g\ 1.$
$\Rightarrow (f +_P g) +_P h$
$\in ((O_R, \{0.; 1.\}, \tau)\ HomotopyClass)$
$(f +_P g +_P h)$

path_plus_0_thm

$\vdash \forall\ \tau\ f$
$\bullet\ \tau \in Topology \wedge f \in Paths\ \tau$
$\Rightarrow f +_P 0_P\ (f\ 1.)$
$\in ((O_R, \{0.; 1.\}, \tau)\ HomotopyClass)\ f$

path_0_plus_thm

$\vdash \forall\ \tau\ f$
$\bullet\ \tau \in Topology \wedge f \in Paths\ \tau$
$\Rightarrow 0_P\ (f\ 0.) +_P f$
$\in ((O_R, \{0.; 1.\}, \tau)\ HomotopyClass)\ f$

path_plus_minus_thm

$\vdash \forall\ \tau\ f$
$\bullet\ \tau \in Topology \wedge f \in Paths\ \tau$
$\Rightarrow f +_P \sim_P f$
$\in ((O_R, \{0.; 1.\}, \tau)\ HomotopyClass)$
$(0_P\ (f\ 0.))$

path_minus_minus_thm

$\vdash \forall\ f \bullet \sim_P (\sim_P f) = f$

path_minus_plus_thm

$\vdash \forall\ \tau\ f$
$\bullet\ \tau \in Topology \wedge f \in Paths\ \tau$
$\Rightarrow \sim_P f +_P f$
$\in ((O_R, \{0.; 1.\}, \tau)\ HomotopyClass)$
$(0_P\ (f\ 1.))$

open_connected_path_connected_thm

$\vdash \forall\ \tau\ A$
$\bullet\ \tau \in Topology$
$\wedge\ \tau \in LocallyPathConnected$
$\wedge\ A \in \tau$
$\wedge\ A \in \tau\ Connected$
$\Rightarrow A \in \tau\ PathConnected$

open_interval_path_connected_thm

$\vdash \forall\ x\ y \bullet OpenInterval\ x\ y \in O_R\ PathConnected$

$\mathbb{R}$_locally_path_connected_thm

$\vdash O_R \in LocallyPathConnected$

product_locally_path_connected_thm

$\vdash \forall\ \sigma\ \tau\ f\ a\ b\ c$
$\bullet\ \sigma \in Topology$
$\wedge\ \tau \in Topology$
$\wedge\ \sigma \in LocallyPathConnected$

$\land\ \tau \in \mathit{LocallyPathConnected}$
$\Rightarrow \sigma \times_T \tau \in \mathit{LocallyPathConnected}$

***covering_projection_fibration_thm*1**

$\vdash \forall\ \rho\ \sigma\ \tau\ p\ f\ h$
$\bullet\ \rho \in \mathit{Topology}$
$\land\ \sigma \in \mathit{Topology}$
$\land\ \tau \in \mathit{Topology}$
$\land\ p \in (\sigma,\ \tau)\ \mathit{CoveringProjection}$
$\land\ f \in (\rho,\ \sigma)\ \mathit{Continuous}$
$\land\ h \in (\rho \times_T O_R,\ \tau)\ \mathit{Continuous}$
$\land\ (\forall\ x \bullet\ x \in \mathit{Space}_T\ \rho \Rightarrow h\ (x,\ 0.) = p\ (f\ x))$
$\Rightarrow (\exists\ L$
$\bullet\ L$
$\in ((\mathit{Space}_T\ \rho \times\ \mathit{ClosedInterval}\ 0.\ 1.)$
$\lhd_T\ \rho \times_T O_R,\ \sigma)\ \mathit{Continuous}$
$\land\ (\forall\ x \bullet\ x \in \mathit{Space}_T\ \rho \Rightarrow L\ (x,\ 0.) = f\ x)$
$\land\ (\forall\ x\ s$
$\bullet\ x \in \mathit{Space}_T\ \rho \land s \in \mathit{ClosedInterval}\ 0.\ 1.$
$\Rightarrow p\ (L\ (x,\ s)) = h\ (x,\ s)))$

covering_projection_continuous_thm

$\vdash \forall\ \sigma\ \tau\ p$
$\bullet\ \sigma \in \mathit{Topology}$
$\land\ \tau \in \mathit{Topology}$
$\land\ p \in (\sigma,\ \tau)\ \mathit{CoveringProjection}$
$\Rightarrow p \in (\sigma,\ \tau)\ \mathit{Continuous}$

covering_projection_fibration_thm

$\vdash \forall\ \rho\ \sigma\ \tau\ p$
$\bullet\ \rho \in \mathit{Topology}$
$\land\ \sigma \in \mathit{Topology}$
$\land\ \tau \in \mathit{Topology}$
$\land\ p \in (\sigma,\ \tau)\ \mathit{CoveringProjection}$
$\Rightarrow (\rho,\ p,\ \sigma,\ \tau) \in \mathit{HomotopyLiftingProperty}$

covering_projection_path_lifting_thm

$\vdash \forall\ \sigma\ \tau\ p\ y\ f$
$\bullet\ \sigma \in \mathit{Topology}$
$\land\ \tau \in \mathit{Topology}$
$\land\ p \in (\sigma,\ \tau)\ \mathit{CoveringProjection}$
$\land\ f \in \mathit{Paths}\ \tau$
$\land\ y \in \mathit{Space}_T\ \sigma$
$\land\ p\ y = f\ 0.$
$\Rightarrow (\exists\ g$
$\bullet\ g \in \mathit{Paths}\ \sigma \land g\ 0. = y \land (\forall\ s \bullet\ p\ (g\ s) = f\ s))$

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